tion) was ~ 9 times larger than the average count for LMFE based upon the standard matrix inversion. This is a conservative estimate, as the computational load of LMFE is amenable to further improvement using inversion algorithms that exploit the centrosymmetric character [8] of the regularized modified covariance matrix.

Acknowledgment

The authors would like to acknowledge discussions with Michael D. Zoltowski.

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Synchronizing Chaotic Circuits

Thomas L. Carroll and Louis M. Pecora

***Abstract*—Although the motions of independent chaotic systems are uncorrelated with each other, it is possible under some conditions to synchronize a subsystem of one chaotic system with a separate chaotic system by sending a signal from the chaotic system to the subsystem. We describe here the conditions necessary for synchronization and demonstrate synchronization with a chaotic circuit.**

I. Introduction

Chaos is sometimes described as a situation in which a system gets out of synchronization with itself [1], resulting in a complex nonperiodic motion. If two independent chaotic systems are started with the same initial conditions, any arbitrarily small difference in these conditions will grow exponentially in time [2].

Manuscript received April 10, 1990; revised October 18, 1990. The work of T. L. Carroll was supported by an Office of Naval Technology postdoctoral fellowship. This paper was recommended by Associate Editor H. D. Chiang.

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IEEE Log Number 9042418.

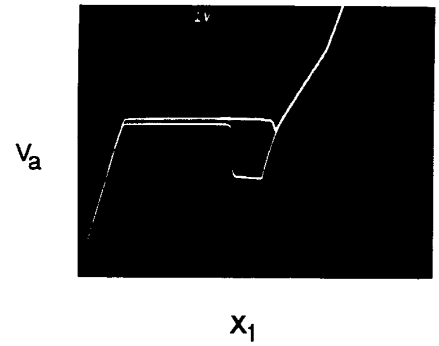

Fig. 1. Voltages x_2 and x_2' are from two identical chaotic circuits running independently of each other.

After some time, the motion of the two systems will be uncorrelated. For example, Fig. 1 shows a plot of a voltage from a chaotic circuit versus a voltage from the same point in another identical but independent circuit. The trajectory is traced randomly over the screen, whereas if the circuits were synchronized, the trace would be a straight 45° line.

It is, however, possible to have two nonlinear systems synchronized, despite their chaotic motion, provided they are both driven with the proper signal. We have devised a way to drive a subsystem of a chaotic system with a signal from a similar chaotic system so that corresponding signals from the two subsystems are identical [3]. We have built a simple circuit based on chaotic circuits described by Newcomb *et al.* [4], [5]. We use this circuit to demonstrate this chaotic synchronization.

II. Theory

The essentials of the theory have been covered elsewhere [3]; we include enough of these here to understand the application to a circuit.

The general scheme for creating synchronizing systems is to take a (nonlinear) system, duplicate some subsystem of this system and drive the duplicate and the original subsystem with signals from the unduplicated part. This is a generalization of "driving" or "forcing" a system. The process can be visualized with ordinary differential equations.

Let $u = f(u)$ be an n-dimensional dynamical system, so that $u = (u_1, \cdots, u_n)$. Divide the system into two subsystems, $v = g(v, w)$ and $w = h(v, w)$, where for a particular value of m, $v = (v_1, \cdots, v_m)$, $g = (g_1, \cdots, g_n)$, $w = (w_1, \cdots, w_{n-m})$, and $h = (h_1, \cdots, h_{n-m})$. Now, duplicate the w subsystem and use as the new variables w'. This yields a $(2n - m)$-dimensional system:

$$\dot{v} = g(v, w)$$
$$\dot{w} = h(v, w)$$
$$\dot{w}' = h(v, w'). \qquad (1)$$

We refer to the $v - w$ subsystem as the drive system since it runs independently of w' and the v signal is fed to the w' subsystem to drive it. We call the w' subsystem the response.

Under the right conditions as time elapses the $w'(t)$ variables will converge asymptotically to the $w(t)$ variables and continue to remain in step with the $w(t)$ values. The necessary and sufficient conditions for this to happen are the signs of the conditional Lyapunov exponents of the w subsystem. The conditional Lyapunov exponents are defined in the following way. In

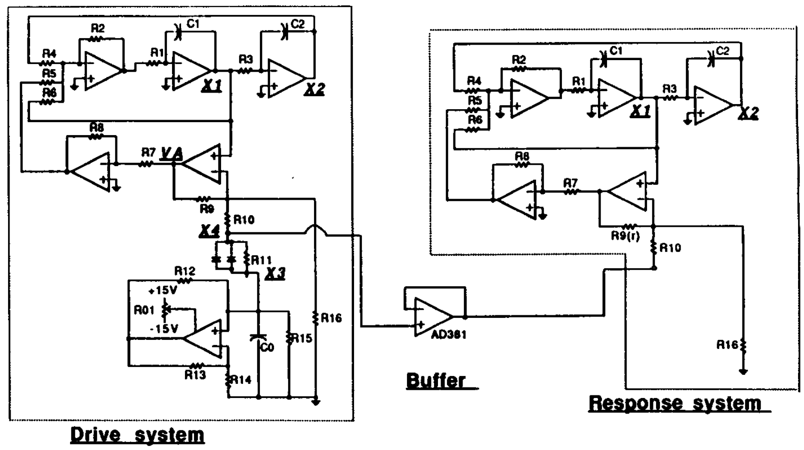

Fig. 2. Circuits used to demonstrate chaotic synchronization. Component values are $R1 = R3 = R7 = R8 = R16 = 100$ kΩ; $R2 = 330$ kΩ; $R4 = 220$ kΩ; $R5 = R6 = 150$ kΩ; $R9 = R15 = 20$ kΩ; $R10 = 3$ kΩ; $R11 = 50$ kΩ; $R12 = R13 = R14 = 10$ kΩ; $R01 = 10$ kΩ pot.; $C0 = C2 = 0.001$ μF; $C1 = 0.01$ μF. The diodes are type 1N4739A. All op amps were type 741 unless otherwise noted. Points labeled $X1$, $X2$, $X3$, and $X4$ are where the corresponding voltages were measured.

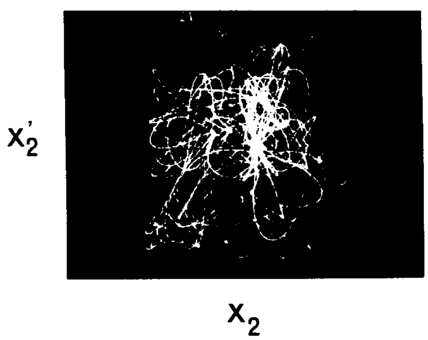

Fig. 3. Response curve for the hysteretic part of the circuit. Input voltage x_1 was supplied by a triangle wave generator operating at 10 Hz. Output voltage V_a was measured at point VA with all feedback disconnected.

order to have w' converge to w, we want $\Delta w(t) = w'(t) - w(t)$ to converge to zero as $t \to \infty$. This leads to the variational equation for the w subsystem

$$\frac{d\Delta w}{dt} = D_w h(v,w)\,\Delta w + O\left((\Delta w)^2\right) \tag{2}$$

where $D_w h$ is the Jacobian of the w subsystem. In the limit of small Δw we can ignore the higher order terms and we are left with a variational equation for the w subsystem.

The Lyapunov exponents resulting from the variational equation can be calculated [3], [6]. They will determine the stability of the w subsystem and thus the convergence of w' to w. Note that these exponents will depend on the driving variables, v, and so are referred to as conditional Lyapunov exponents.

When the exponents are all negative, the w subsystem is stable, which guarantees that w' will not diverge from w. This provides a necessary condition for synchronization. The question that naturally arises is whether there exists a nonempty set of points "nearby" $w(t)$ that will eventually converge to $w(t)$ as $t \to \infty$. The answer is yes and is provided by the linearization principle [7] used in stability theory. This provides the sufficient condition for synchronization.

III. Application to a Circuit

We have used the system shown in Fig. 2 to demonstrate these ideas. The chaotic circuit used is based on those described by Newcomb *et al.* [4], [5]. At the heart of the circuit is an unstable second-degree oscillator that oscillates in the range of several hundred hertz. This oscillator is connected to a hysteretic element. The $I-V$ curve for this element is shown in Fig. 3. The output of the hysteretic element provides a bias that moves the origin for the unstable oscillations to a positive or negative value. The amplitude of the unstable oscillations will increase until the voltage x_1 passes through the switching region of the hysteretic element. This changes the sign of the output of the hysteretic element, shifting the center of oscillation to a new point and keeping the x_1 voltage bounded. Once this center point has been changed, the value of the voltage at x_1 begins to increase, continuing the process. The hysteresis adds the third degree of freedom necessary to produce chaos.

This circuit may be modeled by the set of equations:

$$\begin{aligned}\dot{x}_1 &= x_2 + \gamma x_1 + c x_3\\ \dot{x}_2 &= \omega x_1 - \delta_2 x_2\\ \epsilon \dot{x}_3 &= \left(1 - x_3^2\right)\left(S x_1 - D + x_3\right) - \delta_3 x_3\end{aligned} \tag{3}$$

where $\gamma = R_2/R_6$, $c = R_2/R_5$, $\omega = R_1C_1/(R_2C_3)$, and δ_2 and δ_3 are damping parameters that model losses in the system. The equations for x_1 and x_2 model the unstable second-degree oscillator in the top part of the circuit in Fig. 2. The x_3 equation is based on a function used by Rossler to model hysteresis [8]. The x_3 voltage behaves hysteretically depending on the x_1 voltage. The value of ϵ is small, which causes the x_3 variable to depend on a "faster" time so that the hysteresis quickly reaches the saturation value as the x_1 voltage reaches the trigger point. The S and D variables are derived from the upper and lower hysteresis trigger voltages (V_U and V_L) as measured in the lower part of the circuit in Fig. 2. Thus $S = 2.0/(V_U - V_L)$ and $D = SV_L + 1.0$. For the chaotic regime, $\gamma = 0.2$, $c = 2.2$, $\delta_2 = 0.001$, $\delta_3 = 0.001$, $\epsilon = 0.3$, $\omega = 10$, $S = 1.667$, and $D = 0.0$.

Following the theory developed above, the conditional Lyapunov exponents for a system composed of x_2 and x_3 are -9.98×10^{-4} and -5.01 $(\text{ms})^{-1}$. It should be possible to obtain synchronization by driving the x_2, x_3 subsystem with the x_1 signal. Unfortunately, it is difficult in practice to exactly match the hysteretic elements in the two systems. Driving with the x_1

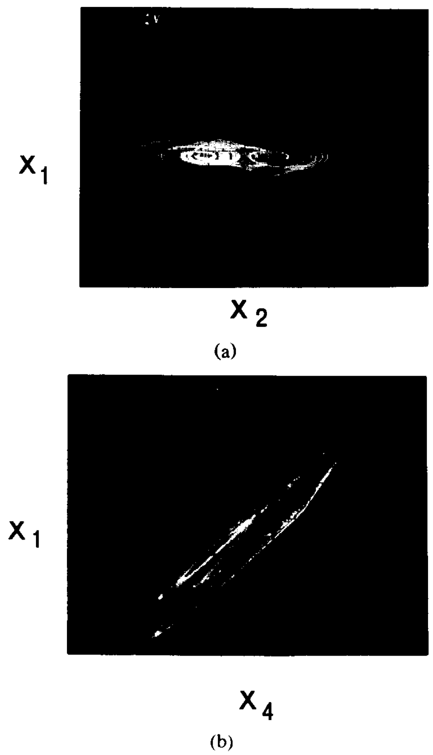

Fig. 4. (a) x_2 versus x_1 voltages for the drive circuit. (b) x_4 versus x_1 voltages.

signal produces poor synchronization.

One may produce a better subsystem by removing part of the hysteretic element. In the system shown in Fig. 2, the driving signal x_4 used is a linear combination of x_3 and x_1.

$$x_4 = \frac{1}{\alpha}(x_3 + \beta x_1) \tag{4}$$

where $\alpha = R_9/R_{10}$ and $\beta = 1 + R_9/R_{02} + R_9/R_{10}$. In terms of x_4, the new equations of motion for this system are

$$\begin{aligned} \dot{x}_1 &= x_2 + \gamma x_1 + c(\alpha x_4 - \beta x_1) \\ \dot{x}_2 &= -\omega x_1 + \delta_2 x_2 \\ \dot{x}_4 &= \frac{1}{\alpha}\Big[\big(1 - (\alpha x_4 - \beta x_1)^2\big)(Sx_1 - D + \alpha x_4 - \beta x_1)/\epsilon \\ &\quad - \delta_3(\alpha x_4 - \beta x_1)/\epsilon - \beta x_2 - \beta\gamma x_1 - c\beta(\alpha x_4 - \beta x_1)\Big]. \end{aligned} \tag{5}$$

In the chaotic regime studied here, $\alpha = 6.67$ and $\beta = 7.87$. The sub-Lyapunov exponents for the x_1, x_2 subsystem are -16.587 ms^{-1} and -0.603 ms^{-1}. The two subcircuits should synchronize.

Fig. 4(a) shows an oscilliscope trace of the x_1 versus x_2 voltage for the driving circuit. Fig. 4(b) shows the x_1 versus x_4 voltage. We pass the x_4 signal through a buffer amplifier and use it to drive a response circuit as shown in Fig. 2. In the response circuit, we can replace the resistor $R_9(r)$ with several different resistors. This allows us to make a response circuit that is identical or not identical to the corresponding part of the drive circuit.

Fig. 5 shows the result of driving the response circuit with a chaotic signal. Fig. 5(a) is a plot showing x_2 for the driving circuit versus x_2 for the response circuit (x_2') with $R_9(r) = R_9 =$ 20 kΩ. In this case, the two circuits are synchronized, so the x_2 signals for the two circuits are the same.

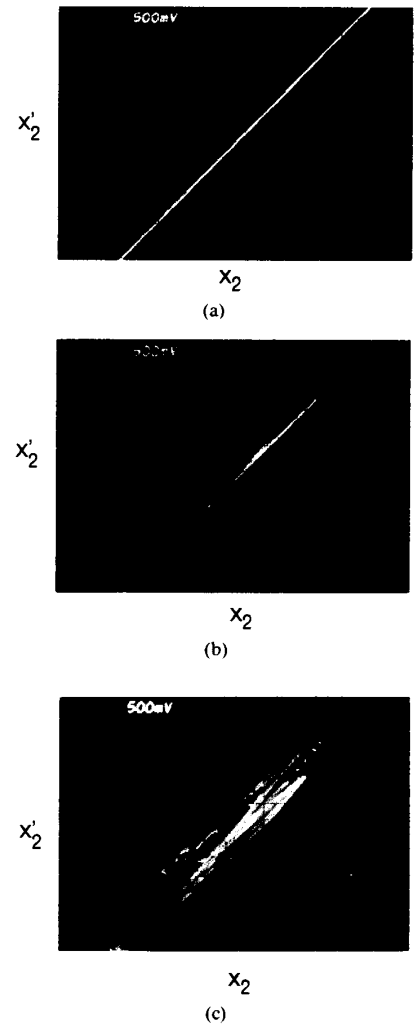

Fig. 5. One of the drive circuit signals x_2 is plotted on the x axis, while the corresponding signal x_2' from the response circuit is plotted on the y axis. (a) $R9(r) = R9 = 20$ kΩ, and the circuits are synchronized. (b) $R9(r) = 22$ kΩ, and the response circuit no longer follows the drive circuit exactly. (c) $R9(r) = 30$ kΩ, and the difference between the two signals is even greater.

Fig. 5(b) shows x_2(drive) versus x_2(response) when $R_9(r) = 22$ kΩ. The two circuits are not quite synchronized, so x_2(drive) is no longer equal to x_2(response). Fig. 5(c) shows x_2(drive) versus x_2(response) when $R_9(r) = 30$ kΩ. The differences between signals in the two circuits are now even larger now that the two systems are even farther from being identical.

One may describe the signals present in each circuit by a trajectory in a phase space, a space in which each axis corresponds to the value of a variable in the circuit, such as x_1, x_2, x_3 or a similar combination. When two circuits are synchronized, corresponding signals in the two circuits are identical. The two systems are at the same point on the same trajectory at the same time. If the two systems are started with different initial conditions, the trajectory of the response system will approach the trajectory of the drive system at an exponential rate in time.

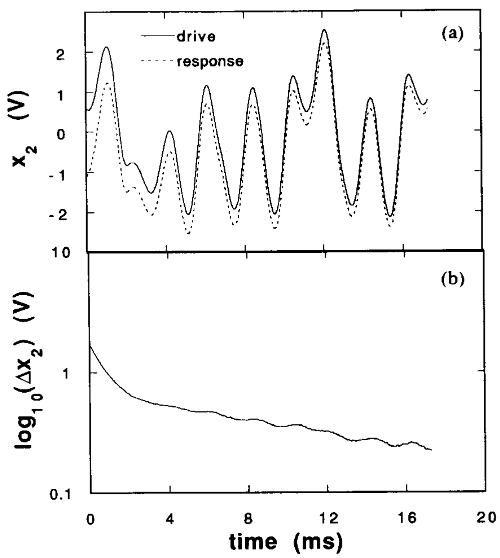

Fig. 6. (a) Drive (solid line) and response (dotted line) signals immediately after synchronization starts. (b) Log base 10 of the difference between these two signals.

This is a consequence of the fact that all the Lyapunov exponents for the response system are negative. Fig. 6 demonstrates this for the circuit described here. To produce this figure, we multiply the drive signal by a ± 1 using a square wave generator and an AD 632 analog multiplier chip. As the square wave switches from -1 to $+1$, we capture the difference signal $x_2(\text{drive}) - x_2(\text{response})$ with a transient digitizer. Fig. 6(a) shows the two signals starting at the point at which synchronization begins. Fig. 6(b) shows the log base 10 of the difference Δx_2 between these two signals. The difference signal has two different exponential decay time constants. The initial decay rate seen is 0.667 ms^{-1}, while after about 4 ms the decay rate is 0.069 ms^{-1}. One expects two decay rates because this is a two-dimensional system. The difference between trajectories is decaying along two different directions in phase space, each with its own rate.

The two decay time constants (1.4 ms and 14.5 ms) are each about a factor of 14 greater than the integrator time constants. Decay rates, integrator time constants, and Lyapunov exponents are all related, but so far we can only find an explicit relation for the latter two.

IV. Conclusion

Computer simulations on the Lorenz and Rossler systems have confirmed that this circuit is not a special case. There are many other circuits that we could have studied [9]; we chose this one for its insensitivity to external influences.

The type of synchronization that we describe above would be trivial for nonchaotic systems, but it is not trivial for chaotic systems. We have built circuits, such as one described by Mitschke and Fluggen [10], in which no synchronization was possible. We could not build a subsystem of this circuit for which all Lyapunov exponents were less than zero.

Acknowledgment

The authors wish to thank Prof. R. W. Newcomb for useful discussions on chaotic circuits.

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The Single CCII Biquads with High-Input Impedance

Shen-Iuan Liu and Hen-Wai Tsao

***Abstract*—Two new configurations for the design of single-CCII-biquad filters with high-input impedance are presented. They can synthesize low-pass, bandpass, high-pass, all-pass, and notch filtering functions with a single CCII connected to five passive one-port *RC* elements. The active and passive sensitivities have been calculated in order to determine the suitable domain for filtering applications. The quality factor Q and natural frequency ω_0 of the proposed SCB's are insensitive to the voltage tracking error of the CCII. Experimental results that confirm theoretical analysis are obtained. The circuits studied here are illustrative of the versatilities of the CCII's.**

I. Introduction

The use of the second generation current conveyors (CCII's) in many applications has been found beneficial [1]–[4]. The advantages of this current mode active element (CCII) have been demonstrated to provide wider bandwidth and better accuracy compared with the conventional op amps [5]–[7]. Therefore, it is desirable to develop various analog signal processing circuits, such as analog filtering circuits, amplifiers, and oscillators using CCII with respect to the counterparts of the op amps. The application and advantage in the synthesis of various active filter transfer functions using CCII's have received considerable attention [8]–[13]. There are many filtering applications that require only a second-order filter in practice, or utilize biquad filters as basic building blocks to synthesize various high-order active filters [14]. Since a high-order filter requires a large

Manuscript received April 17, 1990. This work was supported by the Electronics Research and Service Organization and by the Industrial Technology Research Institute of the Republic of China. This paper was recommended by Associate Editor R. K. Hester.

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IEEE Log Number 9042419.