Abstract

In this paper, the geometric design problem of serial-link robot manipulators with three revolute (R) joints is solved using a polynomial homotopy continuation method. Three spatial positions and orientations are defined and the dimensions of the geometric parameters of the 3-R manipulator are computed so that the manipulator will be able to place its end-effector at these three pre-specified locations. Denavit and Hartenberg parameters and 4x4 homogeneous matrices are used to formulate the problem and obtain eighteen design equations in twenty-four design unknowns. Six of the design parameters are set as free choices and their values are selected arbitrarily. Two different cases for selecting the free choices are considered and their design equations are solved using polynomial homotopy continuation. In both cases for free choice selection, eight distinct manipulators are found that will be able to place their end-effector at the three specified spatial positions and orientations.

1. INTRODUCTION

The calculation of the geometric parameters of a multi-articulated mechanical or robotic system so that it guides a rigid body in a number of specified spatial locations or precision points is known as the Rigid Body Guidance Problem. In this paper, it will also be called the Geometric Design Problem. The precision points are described by six parameters: three for position and three for orientation. This problem has been studied extensively for planar mechanisms and robotic systems and has recently drawn much attention to researchers for spatial multi-articulated systems. Solution techniques for the geometric design problem may be classified into two categories: exact synthesis and approximate synthesis.

Exact synthesis methods result in mechanisms and manipulators, which guide a rigid body exactly through the specified precision points. Solutions in the exact synthesis exist only if the number of independent design equations obtained by the precision points is less than or equal to the number of design parameters. If the number of design equations is less than the number of design parameters, then several of the design parameters can be regarded as free choices and their values can be selected arbitrarily so that a well-determined system is obtained. The number of precision points that may be prescribed for a given mechanism or manipulator is limited by the system type and the number of design parameters that are selected to be free choices [1]. For each manipulator/mechanism there is a specific number of precision points for which without selecting any free choices, there is a finite number of exact solutions to the geometric design problem. This number of precision points depends on the number of design parameters and the type of joints and can be calculated using Tsai and Roth’s formula [2], [3].

In approximate synthesis, using an optimization algorithm, a mechanism is found that, although not guiding a rigid body exactly through the desired poses, it optimizes an objective function defined using information from all the desired poses. Approximate synthesis is mainly used in over-determined geometric design problems where more precision points are defined than required for exact synthesis and therefore no exact solution exists. A complete listing of the extensive amount of research that has been performed in the geometric design of spatial mechanisms and robotic systems, both exact and approximate, can be found in [4].

The equations for the geometric design problem of mechanisms and manipulators are mathematically represented by a set of non-linear, highly coupled multivariate polynomial equations. The solutions of these equations can be obtained by either numerical continuation methods or algebraic methods [5]. Algebraic methods solve the polynomial system by eliminating all but one variable that gives a polynomial equation in one variable. All the solutions are

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then obtained by solving for the roots of the final polynomial. Polynomial continuation is a numerical method that computes all the solutions of a system of polynomials by tracing continuously a finite number of solution paths from a polynomial start system to the final system, which is the polynomial system that we want to solve.

Using algebraic methods, the exact synthesis of planar mechanisms for rigid body guidance has been studied extensively by many researchers and is described in most textbooks on mechanism synthesis [6], [7]. The exact synthesis of a few spatial mechanisms and manipulators has been solved using algebraic methods. The spatial geometric design problems that captured the most attention were the spatial revolute-revolute (R-R) [8]-[12] and the cylindrical-cylindrical (C-C) manipulators [13]-[15]. Other than these two dyads, the geometric design problem has been solved algebraically for the following spatial manipulators/mechanisms. Innocenti [16] solved the geometric design problem for the sphere-sphere binary link. Neilsen and Roth [15] solved the slider-slider sphere dyad, cylinder-cylinder binary link, revolute-slider-sphere dyad and cylinder-sphere binary link design problem. McCarthy [17] also solved the exact synthesis problem for several types of dyads. Even though algebraic methods had been demonstrated to be very effective in solving several geometric design problems for spatial mechanical systems there exist many types of robotic and mechanical systems that are used very often in practical applications, such as the 3R, 4R and 5R manipulators, for which the exact synthesis of the geometric design problem has not been solved before. The main reason for this is the high complexity of the non-linear polynomial design equations that are obtained.

Polynomial continuation methods have been used extensively in the geometric analysis and design of mechanisms and robotic systems [18]. They have been very efficient in solving very difficult problems in the geometric design and analysis of robot manipulators for which no algebraic solution had been discovered. The most representative examples are the continuation solutions of the inverse kinematic problem of the general 6R serial manipulator [19] and the forward kinematic problem of the general parallel platforms [20]. For both problems continuation methods calculated correctly all solutions and they served as targets for the algebraic solutions that followed soon after. Roth and Freudenstein [21] were the first to use continuation methods to solve polynomial systems obtained in the kinematic synthesis of mechanisms. Morgan and Wampler [22] solved the path following design problem of 4R closed loop, planar mechanisms using continuation method when five precision points and the location of the fixed pivots are defined. The same problem was solved with continuation method by Wampler, Morgan and Sommese [23] when nine precision points are specified and no free choices are made. These researchers showed that there are up to 4,326 distinct design solutions for this problem. Dhingra, Cheng and Kohli [24] solved several design problems for six link, slider crank and four-link planar mechanisms using polynomial continuation methods. Polynomial continuation methods have never been used to solve geometric design problems of spatial systems.

In this paper, the geometric design problem of 3R spatial manipulators is solved using polynomial continuation methods. Prior work related to this problem is very limited. Tsai [2] and Roth [9] used screw theory to obtain the design equations for this problem but did not solve them. Ceccarelli [26] used Denavit and Hartenberg parameters to formulate the path generation design equations for spatial 3R manipulators where a number of end-effector positions (not orientations) are specified and used a Newton-Raphson algorithm to solve these equations.

In the problem studied in this paper, three spatial positions and orientations are defined and the dimensions of the geometric parameters of the 3R manipulator are computed so that the manipulator will be able to place its end-effector at these three pre-specified locations. Denavit and Hartenberg (DH) parameters and 4x4 homogeneous matrices are used to formulate the problem and obtain eighteen design equations in twenty-four design unknowns. Six of the design parameters are set as free choices and their values are selected arbitrarily. Two different cases for selecting the free choices are considered and their design equations are solved using polynomial continuation. In both cases for free choice selection, eight distinct manipulators are found that will be able to place their end-effector at the three specified spatial positions and orientations. The polynomial homotopy continuation method is implemented using the software package PHC developed by Verschelde and Cools [27], [28].

2. POLYNOMIAL CONTINUATION

Polynomial continuation is a numerical method that computes all the solutions of a system of polynomial equations by tracing a finite number of solution paths from a polynomial start system to the target system of interest. There are two main steps in using polynomial continuation method: a) generate the start system, b) trace a finite number of solution paths to obtain all the solutions of the polynomial system of interest.

A start system is always generated based on an upper bound on the number of solutions of the target system. There are several known upper bounds, the only one relevant to us is the multi-homogeneous Bezout bound. The computa-
tion of this bound is based on a partition of the variables. Suppose that we have a system of \( n \) equations in \( n \) unknowns \( x_1, \ldots, x_n \). We can partition the \( n \) variables into \( m \) groups using one of the following ways:

\[
G_1 = \{x_{11}, \ldots, x_{1k_1}\}, \ G_2 = \{x_{21}, \ldots, x_{2k_2}\}, \ldots, \ G_m = \{x_{m1}, \ldots, x_{mk_m}\} \text{ with } \bigcup_{i=1}^{m} G_i = \{x_1, \ldots, x_n\}
\]  

(1)

The subscript \( k_i \) is the number of variables in the \( i \)-th group. The maximum degree of the \( j \)-th equation, with respect to the \( i \)-th variable group will be denoted by the parameter \( d_{ij} \). The multi-homogeneous Bezout bound (also known as the \( m \)-homogeneous number) with respect to this partition is defined as the coefficient of the product \( \prod_{i=1}^{m} \alpha_i^{k_i} \) in the polynomial \( \prod_{j=1}^{n} \left( \sum_{i=1}^{m} d_{ij} \alpha_i \right) \) where variables \( \alpha_1, \ldots, \alpha_m \) are introduced for bookkeeping.

Once a partition is specified, a start system with the number of solutions equal to the \( m \)-homogeneous Bezout number can be computed. The general procedure for start system generation can be found in [29].

After generating the start system, the second step is to follow a finite number of solution paths to obtain the solutions of the target system. Suppose that the target system is \( F(x) \) and that the start system is \( G(x) \). Then the polynomial system \( H(x,t) \) is defined as:

\[
H(x,t) = c(1-t)^k G(x) + t^k F(x)
\]

(2)

where \( x = (x_1, \ldots, x_n) \) is a complex \( n \)-tuple, \( t \in [0,1] \), \( c \) is a randomly chosen complex number and \( k \) is a positive integer (usually 1 or 2). The system \( H(x,t) \) is known as linear and quadratic homotopy when \( k \) equals 1 and 2, respectively. From the definition above, \( H(x,0) = 0 \) is a system of \( n \) equations in \( n \) unknowns and an additional parameter \( t \). It is obvious that \( H(x,0) = cG(x) \) and \( H(x,1) = F(x) \). The basic premise of the continuation method is that a small change of \( t \) would result in only a small change of \( x \) as a solution of the system of equations \( H(x,t) = 0 \). That is, if \( x(t) \) is a solution of \( H(x,t) = 0 \), then for a small increment \( \Delta t > 0 \), \( x(t+\Delta t) \) is "near" \( x(t) \). One method of computing \( x(t+\Delta t) \) from \( x(t) \) is to use a predictor-corrector method and the details can be found in [27] and [28]. Each solution of the start system is a solution of \( H(x,t) = 0 \) at \( t=0 \) and represents a solution path of \( H(x,t) = 0 \) from \( t=0 \) to \( t=1 \). Each solution path can be traced independently by successive small increments of \( t \) until \( t \) equals 1. Note that while some of the paths will converge to finite solutions, some others will diverge to solutions at infinity. The solutions of our target system are those that converge finitely in the continuation method as \( t \) approaches 1.

The software used in this research is PHC, a publicly available general-purpose polynomial systems solver using continuation method [27], [28]. Detailed description of polynomial homotopy continuation methods can be found in [18], [27]-[29].

3. PROBLEM FORMULATION

In this work, the relative position of links and joints in mechanisms and manipulators is described using the variant of DH notation that was introduced by Pieper and Roth [30]. In this formulation, the parameters \( a_i, \alpha_i, d_i \) and \( \theta_i \) are defined so that: \( a_i \) is the length of link \( i \), \( \alpha_i \) is the twist angle between the axes of joints \( i \) and \( i+1 \), \( d_i \) is the offset along joint \( i \) and \( \theta_i \) is the rotation angle about joint axis \( i \) as shown in Figure 1. When joint \( i \) is revolute, then \( a_i, \alpha_i \) and \( d_i \) are constants and are called structural parameters, while the value for \( \theta_i \) depends on the configurations and is called the joint variable.

Reference frame \( R_i \) is attached at link \( i \) and its origin \( O_i \) is the intersection point of the common perpendicular between axes \( i \) and \( i-1 \) with joint axis \( i \). Unit vector \( z_i \) of frame \( R_i \) is along joint axis \( i \) unit vector \( x_i \) is along the common perpendicular of joint axes \( i \) and \( i-1 \). Positive directions for \( x_i \) and \( z_i \) are arbitrarily selected. (Note: letters in bold indicate vectors and matrices.) The homogeneous transformation matrix \( A_i \) that describes reference frame \( R_i \) into \( R_{r1} \) and its inverse matrix \( A_i^{-1} \) are found to be equal to:
Consider the three-link open loop spatial chain with revolute (R) joints shown in Figure 2. Two frames are selected arbitrarily: a fixed reference frame $R_0$ and a moving end-effector frame $R_e$. Frame $R_0$ will be defined in three distinct spatial locations. In addition to the three links of the manipulator, a stationary virtual link 0 is also assumed between axis $z_0$ of frame $R_0$ and the first revolute joint axis. Frames are defined at each link using the DH procedure described above. Frame $R_i$ that is stationary is defined attached at link 0 having its $z_i$ axis along the first revolute joint and its $x_i$ axis along the common perpendicular of $z_0$ and $z_i$. Frame $R_{i+1}$ is attached at the tip of link $i$ (where $i=1, 2, 3$). The axis $z_4$ is coincident with the axis $z_c$ of the end-effector frame. The axis $x_i$ is defined along the common perpendicular of $z_i$ and $z_c$ and the origin $O_i$ of $R_d$ is the point of intersection of $z_c$ with its common perpendicular with $z_i$. So frames $R_d$ and $R_e$ have the same z-axis.

The homogeneous transformation matrices $A_i$, with $i=0, 1, 2, 3$ describe frame $R_{i+1}$ relative to $R_i$. The homogeneous transformation matrix $A_e$ relates $R_e$ to $R_0$. The relationship between these frames is a screw displacement: a rotation $\phi$ around the $z_d$ axis and a translation $d$ along the $z_d$ axis. Homogeneous transformation matrix $A_h$ relates directly the end-effector reference frame $R_e$ to the frame $R_0$. Matrices $A_e$ and $A_h$ are written as:

$$
A_e = \begin{pmatrix}
    c_z & -s_z c_{\theta_e} & s_z s_{\theta_e} & a_{\theta_e} \\
    s_z & c_z c_{\theta_e} & -c_z s_{\theta_e} & a_{\theta_e} \\
    0 & s_{\theta_e} & c_{\theta_e} & d_i \\
    0 & 0 & 0 & 1
\end{pmatrix}
$$

$$
A_h = \begin{pmatrix}
    c_d & -s_d & 0 & -a_d \\
    s_d & c_d & 0 & a_d \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
\end{pmatrix}
$$

where $c_z = \cos(\theta_z)$, $s_z = \sin(\theta_z)$, $c_{\theta_e} = \cos(\theta_e)$ and $s_{\theta_e} = \sin(\theta_e)$.

The elements of matrix $A_h$, $A_e$ is known since they represent the position and orientation of frame $R_e$ in $R_0$. The parameters $x_d, y_d,$ and $z_d$ are the coordinates of the origin of $R_e$ in $R_0$.

An important feature in the matrix definition above is the use of matrix $A_c$. In general, six parameters are needed to describe one reference frame relative to another. The DH parameterization succeeds in using four parameters for the relative transformation between frames within the serial kinematic chain itself only after the various motion axes are fixed. However, a special treatment is required, either at the origin or at the end-effector of the serial chain, the latter case being used in this paper. Assuming directions for axes $z_1$, $z_2$ and $z_3$ relative to the fixed reference frame, then the displacement described by the product of matrices $A_0A_1A_2$ can be treated as a displacement of the fixed reference frame to the location of frame $R_0$. At this stage, a general six-parameter displacement is needed to transform frame $R_0$ into the end-effector frame $R_e$. The transformations described by the matrices $A_3$ and $A_e$ provide the complete set of six parameters.

The loop closure equation of the manipulator is used to obtain the design equations:

$$
A_0A_1A_2A_3A_e = A_h
$$

Equation (5) is a 4 by 4 matrix equation that results in six independent scalar equations. The right side of Equation (5), i.e. the elements of matrix $A_h$, is known since they represent the position and orientation of frame $R_e$ at each precision point. The left side of Equation (5) contains all the unknown geometric parameters of the manipulator which are the DH parameters $a_i$, $\alpha_i$, $d_i$ and $\theta_i$ for $i=0, 1, 2, 3$, and parameters $\phi$ and $d$ of matrix $A_c$. Joint angles $\theta_1$, $\theta_2$ and $\theta_3$ have a different value for each precision point while all other 15 geometric parameters are constant. Thus for five precision points there are 30 unknown parameters in total, and there are 30 scalar equation that are obtained. Therefore, the maximum number of precision points for exact synthesis is five. For three precision points synthesis, which is studied in this paper, there are 24 unknowns (15 structural parameters and 9 joint variables) and 18 scalar equations, thus we can select six structural parameters arbitrarily as free choices.

Due to the arbitrary selection of the positive direction of $z_i$ there will be two values for the twist angle, i.e. $\alpha_i$ and $\alpha_i+\pi$, that correspond to the same joint axes $i$ and $i+1$. Similarly, due to the arbitrary selection of the positive direction of $x_i$, there will be two values for the joint angle, i.e. $\theta_i$ and $\theta_i+\pi$, that describes the angle between $x_i$ and $x_{i+1}$. The consequence is that in this problem, where angles $\alpha_i$ and $\theta_i$ are calculated, both values for each one of these parameters will appear among the set of solutions. Obviously, only one of these values will be retained because they correspond to the same set of axes.
4. DESIGN EQUATIONS AT EACH PRECISION POINT

Using the loop closure equation of the manipulator (Equation 5), six scalar design equations are obtained at each precision point. The unknowns in these equations are the manipulator constant structural parameters and the joint variables $\theta_1$, $\theta_2$ and $\theta_3$, which vary from precision point to precision point. To simplify the solution process, we eliminate the joint variables from the design equations at each precision point. Once the joint variables are eliminated, the new set of equations contains only unknowns that do not change from precision point to precision point. In this way, for each new precision point that is defined, new equations are added that have exactly the same form as for the first precision point. In this section we present the method to obtain design equations devoid of the joint variables.

From Equation (3), it can be seen that the 3rd and 4th columns of matrix $A_i^{-1}$ are independent of joint angle $\theta_i$. Therefore, if Equation (5) is written as:

$$A_1 A_2 = A_0^{-1} A_h A_c^{-1} A_3^{-1}$$  \hspace{1cm} (6)$$

then the scalar equations that are obtained by equating the left and right side of the third and fourth columns of matrix Equation (6) will be devoid of joint angle $\theta_3$.

From the third column of Equation (6), three scalar equations are obtained:

$$s\alpha_2 c1s - c\alpha_2 s1c + s\alpha_1 c\alpha_2 s1 = c0L1 + s0L2$$  \hspace{1cm} (7)$$

$$s\alpha_1 c2s - c\alpha_1 s1c - s\alpha_2 c\alpha_1 q = c\alpha s0L1 + c\alpha c0L2 + s\alpha c\alpha L3$$  \hspace{1cm} (8)$$

$$s\alpha_2 s\alpha c2 + c\alpha_2 c\alpha c = s\alpha s0L1 - s\alpha c0L2 + c\alpha c\alpha L3$$  \hspace{1cm} (9)$$

where $L_i = l_i A + m_i B + n_i C$, with $i = 1, 2, 3$, and $A = s\phi \alpha_3$, $B = c\phi \alpha_3$ and $C = c\alpha_3$ with $c\phi = \cos(\phi)$ and $s\phi = \sin(\phi)$.

From the fourth column of Equation (6), another three scalar equations are obtained:

$$a_2 c_2 a_2 c_0 s_0 s_2 + a_1 c_1 + d_0 s_0 s_1 = c_0 (M_1 + x) + s_0 (M_1 + y) - a_0$$  \hspace{1cm} (10)$$

$$a_2 c_2 a_1 c_0 c_0 c_0 s_1 - d_0 s_0 c_1 + a_1 s_1 = -c_0 s_0 s_0 (M_1 + x) + c_0 c_0 (M_1 + y) + s_0 s_0 (M_1 + z) - d_0 s_0$$  \hspace{1cm} (11)$$

$$a_2 s_0 c_2 + d_0 c_2 c_0 s_1 = a_0 c_0 (M_2 + x) - s_0 c_0 (M_2 + y) + c_0 (M_2 + z) - d_0 c_0$$  \hspace{1cm} (12)$$

where $M_i = l_i P + m_i Q + n_i R$, with $i = 1, 2, 3$ and $P = c_3 s_3 s_3 s_3 \phi$, $Q = a_3 s_3 c_3 s_3 \phi$ and $R = -d_3 c_3 \phi - d_3$.

Note that Equations (9) and (12) are free of $\theta_1$, thus, $c_2$ and $s_2$ can be computed by these two equations and their analytical expressions are free of $\theta_1$ also. Using this result, $\theta_2$ is essentially eliminated, for $c_2$ and $s_2$ can be eliminated from any equation by substituting the above result.

The final step is to obtain equations free of $\theta_1$. To obtain such equations, we will consider the matrix Equation (6) again, written here as,

$$A_L = A_R$$  \hspace{1cm} (13)$$

where $A_L = A_1 A_2$ and $A_R = A_0^{-1} A_h A_c^{-1} A_3^{-1}$.

We will denote the third column vector of $A_L$ and $A_R$ as $U_L$ and $U_R$, respectively, and the fourth column vector of $A_L$ and $A_R$ as $V_L$ and $V_R$, respectively (Note: vectors $U_L$, $U_R$, $V_L$ and $V_R$ are 3 by 1: i.e. we neglect the fourth component which is the homogeneous coordinate). Then, we form the following three vector equations:

$$U_L \cdot V_L = U_R \cdot V_R$$  \hspace{1cm} (14)$$

$$V_L \cdot V_L = V_R \cdot V_R$$  \hspace{1cm} (15)$$

$$U_L \times V_L = U_R \times V_R$$  \hspace{1cm} (16)$$

Equations (14), (15) and (16) were originally proposed by Raghavan and Roth to solve the inverse kinematics problem of general six degree of freedom serial link manipulators [5]. The same equations are used here for the geometric design of 3R manipulators.

Equations (14), (15) and (16) give a total of five scalar equations. For Equation (16), only the third component is used, i.e.
It was found that Equations (14), (15) and (17) are naturally devoid of $\theta_1$. With $\theta_2$ eliminated by using the expressions of $c_2$ and $s_2$, calculated from Equations (9) and (12), the three equations are free of $\theta_1$, $\theta_2$ and $\theta_3$ and have, respectively, the following form:

$$\sum_{x_i \in W} f_{x_i x_i} (\alpha_0, \theta_0 \alpha_1) X_j X_k = 0$$  \hspace{1cm} (18)

$$\sum_{x_i \in W} g_{x_i x_i} (\alpha_0, \theta_0 \alpha_1) X_j X_k = 0$$  \hspace{1cm} (19)

$$\sum_{x_i \in W} h_{x_i x_i} (\alpha_0, \theta_0 \alpha_1) X_j X_k = 0$$  \hspace{1cm} (20)

Where $W=\{F, G, H, S, P, Q, R, d_2, a_0, c_0, d_0, d_1, 1\}$ and $F=\lambda A$, $G=\lambda B$, $H=\lambda C$, $S=\lambda c_1\alpha_2$ and $\lambda=a_2/s\alpha_2$.

Note that Equations (18), (19) and (20) depend also on the parameters $l_i$, $m_i$, $n_i$, ($i=1, 2, 3$) and $x$, $y$ and $z$, which are defined at each precision point and vary from precision point to precision point. Therefore, each precision point contributes three design equations (i.e. Equation (18), (19) and (20)), which are devoid of the joint variables and have as unknowns only the 15 constant structural parameters.

5. SOLUTION PROCEDURE USING POLYNOMIAL CONTINUATION

In this paper we solve the geometric design problem of 3R manipulators using three precision points. In this case, there are 9 scalar equations in 15 unknowns. This means that we can select six design parameters as free choices so that a well-determined system of nine equations in nine unknowns is obtained. In this paper, two different ways for selecting free choices have been considered. The first type of free choice selection involves six parameters related to the manipulator base and the first link. For the second type of free choice selection, four free choices are made at the base and the remaining two at the end-effector. The design equations for both types of selections are obtained by substituting the free choices made into Equations (18), (19) and (20).

A. Type 1 of Free Choice Selection

In this type, the free choices made are parameters $d_0$, $a_0$, $\theta_0$, $d_1$ and $a_1$. By arbitrarily selecting the values for these parameters the designer selects the location of the first joint of the manipulator with respect to a fixed reference frame and also selects the first link’s offset and link length. After substituting the values of the free choices into Equations (18), (19) and (20), they become:

$$\sum_{x_i \in T_1} f_{x_i x_i} (\alpha_i) X_j X_k = 0$$  \hspace{1cm} (21)

$$\sum_{x_i \in T_1} g_{x_i x_i} (\alpha_i) X_j X_k = 0$$  \hspace{1cm} (22)

$$\sum_{x_i \in T_1} h_{x_i x_i} (\alpha_i) X_j X_k = 0$$  \hspace{1cm} (23)

where: $T_1=\{F, G, H, S, P, Q, d_2, 1\}$.

Note that $f$, $g$, $h$ are polynomial functions of $c\alpha_1$ and $s\alpha_1$, which are transcendental functions in $\alpha_1$. We can regard $c\alpha_1$ and $s\alpha_1$ as independent variables with an additional constraint equation $c\alpha_1^2 + s\alpha_1^2 - 1 = 0$. By incorporating the constraint equations, the new system is a multivariate polynomial system with ten equations in ten unknowns from $T_1 \cup \{c\alpha_1, s\alpha_1\}$.

Using a 2-partition $G_1=\{F, G, H, S, P, Q, R, d_2\}$ and $G_2=\{c\alpha_1, s\alpha_1\}$, the multi-homogeneous bound is found to be 6144. This bound can be further reduced by a linear reduction of the system which involves subtracting Equations (21), (22), (23) at the second and third precision points by the corresponding equations at the first precision point. After the reduction, using the same 2-partition, the 2-homogeneous number is 448.

Using PHC, a continuation method based on this 2-homogeneous number is employed and the numerical values of the variables in $G_1 \cup G_2$ are computed. The DH parameters of the design solutions are computed using a back-
substitution procedure, which is outlined in the Appendix. It is found that out of 448 paths, only 16 paths converge to true solutions of the design problem, the remaining 432 paths are solutions at infinity and extraneous solutions. These 16 solutions are all numerically different but contain only 8 geometrically distinct solutions, where each geometrically distinct solution has two equivalent different representations in terms of DH parameters (see last paragraph of Section 3).

Therefore, at the most there are eight distinct manipulators that can place their end-effectors in the three precision points specified by the designer when the location of the first joint is selected with respect to a reference frame and when the first link’s offset and link length are specified.

B. Type 2 of Free Choice Selection

In this type, the free choices made are \( d_0, \alpha_0, a_0, \theta_0, d \) and \( \phi \). In this case, the designer selects the location of the first joint of the manipulator with respect to a fixed reference frame. He/she also selects the parameters related to the end-effector geometry.

The design equations (18), (19) and (20) become:

\[
\sum_{x_i \in T_2} f_{x_i} X_i X_k = 0 \tag{24}
\]

\[
\sum_{x_i \in T_2} g_{x_i} X_i X_k = 0 \tag{25}
\]

\[
\sum_{x_i \in T_2} h_{x_i} X_i X_k = 0 \tag{26}
\]

where \( T_2 = \{ U, V, W, H, S, a_1, a_3, d_1, d_2, 1 \} \) with \( U = d_3 s_3 \alpha_3, V = -d_3 c_3 \alpha_3 \) and \( W = \lambda s_3 \alpha_3 \). Equations (24) to (26) represent a total of nine equations in ten unknowns, therefore, an additional equation is needed to have a well-determined system. This additional equation that comes from the interrelations among the variables in \( T_2 \) and from the definitions given, has the following form:

\[
VW + HU = 0 \tag{27}
\]

Together with the constraint equation \( c \alpha_1^2 + s \alpha_1^2 - 1 = 0 \), the new system is a multivariate polynomial system of eleven equations in eleven unknowns \( T_2 \cup \{ c \alpha_1, s \alpha_1 \} \).

Using a 2-partition \( G_1 = \{ U, V, W, H, S, a_1, a_3, d_1, d_2 \} \) and \( G_2 = \{ c \alpha_1, s \alpha_1 \} \), after linear reduction, the 2-homogeneous number is 10240.

Using PHC, a continuation method based on this 2-homogeneous number is used and the numerical values of the variables in \( G_1 \cup G_2 \) are computed. The back-substitution procedures for computing the DH parameters of the design solutions are again given in the Appendix. It is found that out of 10240 paths, only 32 paths converge to true solutions of the design problem and the remaining paths are solutions at infinity and extraneous solutions. As in type 1, these 32 solutions are all numerically different but contain only 8 geometrically distinct solutions, where each geometrically distinct solution has four different but equivalent representations in terms of DH parameters (see also last paragraph in Section 3.)

Therefore, at the most there are eight distinct manipulators that can place their end-effectors in the three precision points specified by the designer when the locations of the first joint is selected with respect to a reference frame and when end-effector geometrical parameters are specified.

6. NUMERICAL EXAMPLE

In this section, one numerical example is shown for each of the two types of free choices. The computation is carried out using the software PHC run on the Sun Microsystems Enterprise 10000 system of the Rutgers University, Center for Advanced Information Processing (CAIP). The Enterprise 10000 is a parallel processing system composed of sixty-four 400 MHz SPARC processors. PHC is a general-purpose polynomial equations solver with continuation method developed by Prof. Verscheide and which is publicly available [27], [28]. The start system is generated by a random linear products based on the multi-homogeneous Bezout number and the path tracking is carried out with a quadratic homotopy.
For both types of free choice selection, three precision points are arbitrarily selected. These precision points are defined by the position coordinates of the origin of the end-effector frame with respect to the fixed reference frame and the direction cosines of the end-effector frame with respect to the fixed reference frame. These three precision points that are selected give the following $A_{hi}$ matrices where $i=1, 2, 3$:

$$A_{h1} = \begin{bmatrix}
-0.866025 & -0.433013 & 0.25 & 2.63397 \\
0.433013 & -0.899519 & -0.0580127 & 8.78109 \\
0.25 & 0.0580127 & 0.966506 & 4.45096 \\
0 & 0 & 0 & 1
\end{bmatrix},$$

$$A_{h2} = \begin{bmatrix}
-0.707107 & -0.5 & 0.5 & 4 \\
0.5 & -0.853553 & -0.146447 & 8.24264 \\
0.5 & 0.146447 & 0.853553 & 5.41421 \\
0 & 0 & 0 & 1
\end{bmatrix},$$

$$A_{h3} = \begin{bmatrix}
0 & 0 & 1 & 6 \\
0 & -1 & 0 & 6 \\
1 & 0 & 0 & 6 \\
0 & 0 & 0 & 1
\end{bmatrix}.$$

For type 1, the numerical values of the free choices made are $d_0=a_0=d_1=a_1=2$ and $\alpha_0=\theta_0=\phi_0=\pi/2$. The computed values of the DH parameters of the eight geometrically distinct manipulators solutions are given in Table 1. The computation is carried out using 15 digits but only 6 digits are shown here. The average time required to complete the computation is 2 hours. Figure 3 shows CAD drawings of the 4 real 3R manipulators obtained as solutions in this case.

For type 2, the numerical values of the free choices made are $d_0=a_0=d=2$ and $\alpha_0=\theta_0=\phi=\pi/2$. The three precision points homogeneous matrices are identical to those given in the example in type 1. The computed values of the eight geometrically distinct design solutions are given in Table 2. The average time required to complete the computation is 70 hours. Figure 4 presents CAD drawings of the 4 real 3R manipulators obtained as solutions in this case.

In Tables 1 and 2, the units for angular parameters are in radians while the units for lengths can be anything provided that it is consistent with the units used to define the end-effector position coordinates shown in the 4th column of matrices $A_{hi}$. Note that I is the square root of –1. In both examples there are only 4 real manipulators that could place their end-effectors at the three specified precision points while the other four solutions are complex.

### 7. CONCLUSIONS

In this paper, the geometric design problem of serial-link spatial robot manipulators with three revolute (R) joints is solved using a polynomial homotopy continuation method. This is the first time that this problem is solved. Three spatial positions and orientations are defined and the DH parameters of the 3-R manipulator are computed so that the manipulator will be able to place its end-effector at these three pre-specified locations. Two types of free choices selections are considered. It is shown that for both types of free choice selection eight manipulators can be found at the most that can place their end-effectors at the three specified precision points. Our current and future work is directed towards the application of this formulation and solution methodology in solving the geometric design problem of spatial 3R chains with four and five precision points. We also work towards developing algebraic solutions for the problem studied in this paper.

### 8. ACKNOWLEDGEMENTS

This work was supported by a National Science Foundation CAREER Award to Professor Mavroidis under the grant DMI-9984051. Mr. Eric Lee was supported by a Computational Sciences Graduate Fellowship from the Department of Energy. The authors would like to thank Dr. Charles Wampler of General Motors and Dr. Jan Verschelde of the University of Illinois at Chicago, for providing helpful suggestions in using polynomial continuation methods and assisting with the use of the software PHC.

### 9. REFERENCES


10. APPENDIX

Once the numerical values of the variables in $G_1 \cup G_2$ are calculated using a polynomial continuation method (see Section 5), then the manipulator DH parameters of the design solutions have to be computed using back-substitution procedures. These procedures are outlined here.

For type 1 free choice selection, the continuation method gives the numerical values of $F, G, H, S, P, Q, R, d_2, c\alpha_1$ and $s\alpha_1$, can be easily found from the value of using the two arguments inverse tangent. The angles $\alpha_2, \alpha_3, \phi$ and the length $a_2$ can be found using the following equations:

$$
\phi = \arctan \left( \frac{F}{G} \right), \quad \alpha_1 = \arctan \left( \frac{F}{H \sin(\phi)} \right), \quad \alpha_2 = \arccos \left( \frac{S \cos(\alpha_1)}{H} \right), \quad a_2 = \frac{H \sin(\alpha_2)}{\cos(\alpha_1)}
$$

By definition, $P, Q$ and $R$ are linear in terms of $a_3, d_3$ and $d$, therefore, after substituting the computed values of $\alpha_3$ and $\phi$ into the definitions of $P, Q$ and $R$, $a_1, d_3$ and $d$ can be found by solving a $3 \times 3$ linear system. The joint angle $\theta_2$ at each of the three precision points is computed by using Equations (9) and (12). Joint angle $\theta_1$ at each precision point is computed by solving a $2 \times 2$ linear system from Equation (7) and (10). Finally, $\theta_3$ is computed by using the first and second column of the matrix Equation (13).

For type 2, the continuation method gives the numerical values of $U, V, W, H, S, a_1, a_3, d_1, d_2, c\alpha_1$ and $s\alpha_1$. By definition, we have the following linear relations:

$$
\begin{bmatrix}
    p \\
    q \\
    r \\
    f \\
    g
\end{bmatrix}
= \begin{bmatrix}
    -\sin(\phi) & 0 & 0 & -\cos(\phi) & 0 \\
    0 & \cos(\phi) & 0 & \sin(\phi) & 0 \\
    0 & 1 & 0 & 0 & -1 \\
    0 & 0 & \sin(\phi) & 0 & 0 \\
    0 & 0 & \cos(\phi) & 0 & 0
\end{bmatrix}
\begin{bmatrix}
    u \\
    v \\
    w \\
    a_3 \\
    d
\end{bmatrix}
$$

We now have the numerical values of $F, G, H, S, P, Q, R, d_2, c\alpha_1$ and $s\alpha_1$, therefore, the back-substitution procedures are identical to that of type 1.
FIGURE 1: Denavit and Hartenberg Parameters

FIGURE 2: Schematic of a 3R Open loop Spatial Manipulator
### TABLE 1: DH Parameters of the 3R Manipulators Found Using Type 1 Free Choices

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<td>-1.00880+3.44639*I</td>
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FIGURE 3: CAD Drawings of the Real 3R Manipulators Found Using Type 1 Free Choices (see Table 1)
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FIGURE 4: CAD Drawings of the Real 3R Manipulators Found Using Type 2 Free Choices (see Table 2)