I. Examples of Loading Effects

A. Electrical System

Consider the following electrical system:

\[ e_i = \frac{1}{C} \frac{d}{ds} e_o \]

\[ e_o = \frac{1}{R + \frac{1}{Cs}} e_i \]

\[ \frac{e_o}{e_i} = \frac{1}{RCS + 1} = G(s) \]

Here we assume that \( l_o = 0 \).

Complete Description:

KCL \( \Rightarrow \frac{d}{ds} l_i + l_o - CE_o = 0 \Rightarrow l_c = -l_o + Cs e_o \)

KVL \( \Rightarrow e_i - Rl_c - e_o = 0 \Rightarrow e_c = e_o + Rl_c \)

\[ \begin{bmatrix} e_o \\ l_o \end{bmatrix} = \begin{bmatrix} RCS + 1 & -R \\ Cs & -1 \end{bmatrix} \begin{bmatrix} e_o \\ l_o \end{bmatrix} \]
Now connect 2 identical RC Low-Pass Filters in series

\[ e_e \rightarrow R \rightarrow R \rightarrow e_o \]

\[ R \quad C \quad C \quad e_e \]

\[ I_s = \frac{e_o}{e_e} \left( \frac{1}{RCs + 1} \right)^2 \quad ? \quad \text{NO!} \]

Let's derive the overall transfer function for the RC-RC Circuit from first principles.
Redraw it to facilitate analysis.
\[
\frac{e_0}{e_A} = \frac{e_0}{e_A} \cdot \frac{e_A}{e_x}
\]

\[
\frac{e_0}{e_A} = \frac{1}{RCs + 1} = \frac{1}{R + \frac{1}{Cs}}
\]

**Z_1:** \( R \) and \( C \) in series

\[
Z_1 = R + \frac{1}{Cs} = \frac{RCs + 1}{Cs}
\]

**Z_2:** \( Z_1 \) and \( C \) in parallel

\[
Z_2 = \frac{(Z_1)(\frac{1}{Cs})}{Z_1 + \frac{1}{Cs}} = \frac{RCs + 1}{(RCs + 2)(Cs)}
\]

\[
\frac{e_A}{e_x} = \frac{Z_2}{R + Z_2} = \frac{RCs + 1}{(RCs + 1)^2 + RCs}
\]

\[
\frac{e_0}{e_x} = \frac{e_0}{e_A} \cdot \frac{e_A}{e_x} = \frac{1}{(RCs + 1)^2 + RCs}
\]

\[
\ne_0 \neq G(s)G(s)
\]

**why?**

\[
\frac{e_0}{e_x} \neq G(s)G(s)
\]
B. Mechanical System

- Consider the following mechanical system:

\[ f_{x_1}, \quad m_1, \quad b_1, \quad b_2, \quad K_2, \quad \dot{x}_{x_2}, \quad \dot{x}_{x_2}, \quad \dot{x}_{o_2}, \quad \dot{x}_{o_2} \]

\[ \chi_{o_1} = \chi_{x_2} \text{ when connected} \]

- Machine-tool slide positioned by a motor position-measuring device to be attached to the slide

- Consider each system separately:

\[ m_1 \ddot{x}_{o_1} + b_1 \dot{x}_{o_1} = f_{x_1} \]

\[ b_2 (\dot{x}_{x_2} - \dot{x}_{o_2}) \]

\[ K_2 \dot{x}_{o_2} \]

\[ (\ddot{x}_{x_2} - \ddot{x}_{o_2})b_2 = K_2 \dot{x}_{o_2} \]

\[ \frac{X_{o_1}}{f_{x_1}} = \frac{1}{m_1 s^2 + b_1 s} \]

\[ \frac{X_{o_2}}{X_{o_1}} = \frac{b_2 s}{b_2 s + K_2} \]

But \( \chi_{o_1} = \chi_{x_2} \) when connected
\[
\frac{\chi_{02}}{f_{x_1}} = G_1(s)G_2(s) = \frac{\frac{b_2}{b_1}K_2}{m_1b_2s^2 + \frac{m_1K + b_1b_2}{b_1K_2}s + 1}
\]

Is This Correct?

Consider the two systems connected:

\[
\begin{align*}
\dot{\chi}_{01} & = b_1\dot{\chi}_{01} + b_2(\dot{\chi}_{01} - \dot{\chi}_{02}) \\
-\frac{m_1}{b_1}\ddot{\chi}_{01} & = f_{x_1} \\
\end{align*}
\]

\[
\begin{align*}
\dot{\chi}_{02} & = b_2(\dot{\chi}_{01} - \dot{\chi}_{02}) + K_2\chi_{02} \\
\end{align*}
\]

\[
\begin{align*}
m_1\ddot{\chi}_{01} + b_1\dot{\chi}_{01} + b_2(\dot{\chi}_{01} - \dot{\chi}_{02}) & = f_{x_1} \\
b_2(\dot{\chi}_{01} - \dot{\chi}_{02}) & = K_2\chi_{02} \\
\end{align*}
\]

Transform and solve for \(\frac{\chi_{02}}{f_{x_1}}\)
\[
\left[ m_1 s^2 + (b_1 + b_2) s \right] x_{01} - (b_2 s) x_{02} = f_{21}
\]

\[ b_2 s x_{01} = (b_2 s + k_2) x_{02} \]

Solve for \( \frac{x_{02}}{f_{u1}} \)

\[ \frac{x_{02}}{f_{u1}} = \frac{b_2}{(b_1 + b_2) k_2} \]

\[ m_1 b_2 s^2 + \frac{m_1 k_2 + b_1 b_2}{(b_1 + b_2) k_2} s + 1 \]

\[ G(s) \neq G_1(s) G_2(s) \]
C. Hydraulic System

\[ g_e - g_o = A \cdot h \]
\[ h = R \cdot g_o \]

Transform and solve for \( \frac{g_o}{g_e} \):

\[ g_e - g_o = A \cdot h \]
\[ h = R \cdot g_o \]

\[ \frac{g_o}{g_e} = \frac{1}{RA_1s + 1} \]

Now consider two tanks interconnected:

\[ Is \quad \frac{g_o}{g_e} = \frac{1}{(RA_1s + 1)(RA_2s + 1)} \]
Write down equations of motion:
\[
g_x - g = A_1 h_1 \\
h_1 - h_2 = R_1 g \\
g - g_0 = A_2 h_2 \\
h_2 = R_2 g_0
\]

Transform and solve for \( \frac{g_0}{g_x} \)

Result:
\[
\frac{g_0}{g_x} = \frac{1}{(R_1 A_2 R_2 A_2) s^2 + (R_1 A_1 + R_2 A_2 + R_2 A_1) s + 1}
\]

\[
= G(s)
\]

\[
G_1(s) = \frac{1}{R_1 A_1 s + 1} \\
G_2(s) = \frac{1}{R_2 A_2 s + 1}
\]

\[
G(s) \neq G_1(s) G_2(s)
\]
II. Impedance

- When a second device is coupled to a first device at its output, it will draw some power from the first. Definition of this power is impossible in terms of a single variable. Two variables are required.

- A two-port device is one that exchanges energy with others at only two locations (ports).

\[
\begin{align*}
\begin{array}{c}
\text{2-port device 1} \\
 e_{21} & \rightarrow & f_{01} & \rightarrow & f_{12} & \rightarrow & f_{02} \\
 e_{01} & \leftarrow & e_{12} & \leftarrow & e_{02}
\end{array}
\end{align*}
\]

\( e \equiv \text{effort variable} \)

\( f \equiv \text{flow variable} \)
<table>
<thead>
<tr>
<th>Class of System</th>
<th>Effort Variable</th>
<th>Flow Variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electrical</td>
<td>Voltage</td>
<td>Current</td>
</tr>
<tr>
<td>Mechanical</td>
<td>Force</td>
<td>Velocity</td>
</tr>
<tr>
<td>Hydraulic</td>
<td>Pressure</td>
<td>Flow Rate</td>
</tr>
<tr>
<td>Thermal</td>
<td>Temperature</td>
<td>Heat-Flow Rate</td>
</tr>
</tbody>
</table>

The product of the two variables at each port (effort $\times$ flow) gives the instantaneous power flowing through the port.

\[
\begin{bmatrix}
e_i \\
\dot{f}_i
\end{bmatrix}
= \begin{bmatrix}
G
\end{bmatrix}
\begin{bmatrix}
e_o \\
\dot{f}_o
\end{bmatrix}
\]

$G = 2 \times 2$ Transfer Function

Assumption: Linear Model for device
Let's derive this Transfer Function Matrix for the 3 Systems discussed.

**RC Circuit**

\[
\begin{align*}
KCL & \Rightarrow i_u + i_0 - Ce_0 = 0 \Rightarrow i_u = -i_0 + Cse_0 \\
KVL & \Rightarrow e_u - Ri_u - e_0 = 0 \Rightarrow e_u = e_0 + Ri_u
\end{align*}
\]

\[
\begin{bmatrix}
e_u \\
i_u
\end{bmatrix} =
\begin{bmatrix}
Rcs + 1 & -R \\
cs & -1
\end{bmatrix}
\begin{bmatrix}
e_0 \\
i_0
\end{bmatrix}
\]

**Mechanical System**

\[
\begin{align*}
\begin{bmatrix}
f_u \\
x_u
\end{bmatrix} &=
\begin{bmatrix}
1 & m_1s^2 + b_1s \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
f_0 \\
x_0
\end{bmatrix}
\end{align*}
\]

\[
f_u - f_0 - b_1 x_0 - m_1 \ddot{x}_0 = 0 \quad x_u = x_0
\]
\[ f_1 - f_0 - K_2 \chi_0 = 0 \quad \Rightarrow \quad f_1 = f_0 + K_2 \chi_0 \]

\[ b_2 (\ddot{x}_0 - \dot{x}_0) - f_0 - K_2 \chi_0 = 0 \quad \Rightarrow \]

\[ b_2 \chi_x = b_2 \chi_0 - K_2 \chi_0 + f_0 \]

\[
\begin{bmatrix}
 f_1 \\
 \chi_1 
\end{bmatrix} = \begin{bmatrix}
 1 & K_2 \\
 \frac{1}{b_2 s} & \frac{b_2 s + K_2}{b_2 s}
\end{bmatrix} \begin{bmatrix}
 f_0 \\
 \chi_0 
\end{bmatrix}
\]

The overall transfer function matrix is:

\[
\begin{bmatrix}
 f_{11} \\
 \chi_{11} 
\end{bmatrix} = \begin{bmatrix}
 1 & m_1 s^2 + b_1 s \\
 0 & 1
\end{bmatrix} \begin{bmatrix}
 1 & K \\
 \frac{1}{b_2 s} & \frac{b_2 s + K_2}{b_2 s}
\end{bmatrix} \begin{bmatrix}
 f_{02} \\
 \chi_{02} 
\end{bmatrix}
\]

\[
\begin{bmatrix}
 f_{11} \\
 \chi_{11} 
\end{bmatrix} = \begin{bmatrix}
 \frac{m_1 s^2 + b_1 s}{b_2 s} + 1 & K + \frac{(m_1 s^2 + b_1 s)(b_2 s + K_2)}{b_2 s} \\
 \frac{1}{b_2 s} & \frac{b_2 s + K_2}{b_2 s}
\end{bmatrix} \begin{bmatrix}
 f_{02} \\
 \chi_{02} 
\end{bmatrix}
\]
Hydraulic System

\[ h_{x1} - h_{o1} = R_1 g_{o1} \]
\[ g_{x1} - g_{o1} = A_1 h_{x1} \]

\[ \begin{bmatrix} h_{x1} \\ g_{x1} \end{bmatrix} = \begin{bmatrix} 1 & R_1 \\ A_1 s & A_1 R_1 s + 1 \end{bmatrix} \begin{bmatrix} h_{o1} \\ g_{o1} \end{bmatrix} \]

Similarly

\[ \begin{bmatrix} h_{x2} \\ g_{x2} \end{bmatrix} = \begin{bmatrix} 1 & R_2 \\ A_2 s & A_2 R_2 s + 1 \end{bmatrix} \begin{bmatrix} h_{o2} \\ g_{o2} \end{bmatrix} \]
The overall transfer function matrix is:

\[
\begin{bmatrix}
    h_{x1} \\
    g_{x1}
\end{bmatrix} = \begin{bmatrix}
    1 & R_1 \\
    A_1 s & A_1 R_1 s + 1
\end{bmatrix}\begin{bmatrix}
    1 & R_2 \\
    A_2 s & A_2 R_2 s + 1
\end{bmatrix}\begin{bmatrix}
    h_{02} \\
    g_{02}
\end{bmatrix}
\]

\[
\begin{bmatrix}
    h_{x1} \\
    g_{x1}
\end{bmatrix} = \begin{bmatrix}
    1 + R_1 A_2 s & R_2 + R_1 + A_2 R_1 R_2 s \\
    (A_1 + A_2) s + A_1 A_2 R_1 s^2 & A_1 R_2 s + (A_1 R_1 s + 1)(A_2 R_2 s + 1)
\end{bmatrix}\begin{bmatrix}
    h_{02} \\
    g_{02}
\end{bmatrix}
\]

- In each case we see that we can write:

\[
\begin{bmatrix}
    e_x \\
    f_x
\end{bmatrix} = \begin{bmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
\end{bmatrix}\begin{bmatrix}
    e_0 \\
    f_0
\end{bmatrix}
\]

\[
e_x = a_{11} e_0 + a_{12} f_0
\]

\[
f_x = a_{21} e_0 + a_{22} f_0
\]
• For linear systems, with no internal energy sources, the $a_{ii}$'s are related by:
  $$a_{11}a_{22} - a_{21}a_{12} = 1$$

Therefore it takes only 3 of these quantities to completely describe the terminal behavior of any 2-port device, no matter how complex it might be internally.

• The equations
  $$e_x = a_{11} e_0 + a_{12} f_0$$
  $$f_x = a_{21} e_0 + a_{22} f_0$$

are actually a particular pair of a set of 12 possible equations one could write relating the $e$'s and $f$'s of a given 2-port device.
Since any 2 of the variables may be considered as independent variables, we can write:

\[ e_\alpha = e_\alpha (e_\alpha, f_\alpha) \quad e_0 = e_0 (e_\alpha, f_\alpha) \]
\[ f_\alpha = f_\alpha (e_\alpha, f_\alpha) \quad f_0 = f_0 (e_\alpha, f_\alpha) \]
\[ f_0 = f_0 (e_\alpha, e_0) \quad e_0 = e_0 (e_\alpha, f_0) \]
\[ f_\alpha = f_\alpha (e_\alpha, e_0) \quad f_\alpha = f_\alpha (e_\alpha, f_0) \]
\[ f_0 = f_0 (e_\alpha, f_\alpha) \quad e_\alpha = e_\alpha (f_\alpha, f_0) \]
\[ e_\alpha = e_\alpha (e_\alpha, f_\alpha) \quad e_\alpha = e_\alpha (f_\alpha, f_\alpha) \]

All of these are potentially useful.

For device #1 we can write:

\[ e_{01} = e_{01} (e_{\alpha 1}, f_{01}) \]
\[ e_{01}(s) = \frac{e_{01}(s)}{e_{\alpha 1}(s)} \left[ e_{\alpha 1}(s) + \frac{e_{01}(s)}{f_{01}(s)} \right] f_{01}(s) \]
\[ f_{01}(s) = 0 \]
\[ e_{\alpha 1}(s) = 0 \]
\[
\frac{e_{o1}(s)}{e_{x1}(s)} \bigg|_{f_{01}(s)=0} = \text{Unloaded Transfer Function } W_u(s)
\]

\[
\frac{e_{01}(s)}{f_{01}(s)} \bigg|_{e_{x1}(s)=0} = \text{Generalized Output Impedance } Z_{g\rightarrow 01}(s)
\]

- For device #2 we can write:

\[
e_{x2} = e_{x2}(f_{x2}, f_{02})
\]

\[
e_{x2}(s) = \frac{e_{x2}(s)}{f_{x2}(s)} \bigg|_{f_{x2}(s)=0} + \frac{e_{x2}(s)}{f_{02}(s)} \bigg|_{f_{02}(s)=0}
\]
\[
\frac{e_{a2}(s)}{f_{a2}(s)} \bigg|_{f_{a2}(s) = 0} = \text{Generalized Input Impedance} \quad Z_{g\ a2}(s)
\]

\[
\frac{e_{a2}(s)}{f_{a2}(s)} \bigg|_{f_{a2}(s) = 0} = \text{Generalized Transfer Impedance} \quad Z_{g\ t2}(s)
\]

- Suppose device #2 has no third device connected at its output and \( f_{o2}(s) = 0 \).

  When the 2 devices are connected, \( e_{o1}(s) = e_{a2}(s) \) and \( f_{o1}(s) = -f_{a2}(s) \).
Combine Equations:

\[ e_{o1}(s) = W_u(s) e_{\alpha 1}(s) + Z_{g01}(s) f_{o1}(s) \]
\[ = W_u(s) e_{\alpha 1}(s) + Z_{g01}(s) \left[ \frac{-e_{o2}(s)}{Z_{g12}(s)} \right] \]
\[ = W_u(s) e_{\alpha 1}(s) + Z_{g01}(s) \left[ \frac{-e_{o1}(s)}{Z_{g12}(s)} \right] \]

\[ e_{o1}(s) + \frac{Z_{g01}(s)}{Z_{g12}(s)} e_{o1}(s) = W_u(s) e_{\alpha 1}(s) \]

\[ \frac{e_{o1}(s)}{e_{\alpha 1}(s)} = W_u(s) \left[ \frac{1}{1 + \frac{Z_{g01}(s)}{Z_{g12}(s)}} \right] \]

This equation clearly shows under what conditions we may couple subsystems accurately using the familiar transfer function (here called the unloaded transfer function) method, and what additional subsystem information
(the two impedances) is needed to get accurate coupling when loading is not negligible.

If \( Z_{q12} \gg Z_{q02} \), then the unloaded transfer function is a good approximation to the unloaded transfer function. This approximation may be good over certain ranges of frequency but not others.

- Let's apply this to the 3 examples.

**RC-RC Circuit**

\[
\begin{bmatrix}
E_{o1} \\
I_{o1}
\end{bmatrix} = \begin{bmatrix} RC_s + 1 & -R \\ Cs & -1 \end{bmatrix} \begin{bmatrix} E_{o1} \\
I_{o1}
\end{bmatrix}
\]

\[
E_{o1} = E_{q2} \\
I_{o1} = -I_{q2}
\]

\[
\begin{bmatrix}
E_{o2} \\
I_{o2}
\end{bmatrix} = \begin{bmatrix} RC_s + 1 & -R \\ Cs & -1 \end{bmatrix} \begin{bmatrix} E_{o2} \\
I_{o2}
\end{bmatrix}
\]

\[
I_{o2} = 0
\]
\[
\begin{bmatrix}
  e_{r1} \\
  \eta_{r1}
\end{bmatrix} =
\begin{bmatrix}
  R^2 C^2 s^2 + 3RCS + 1 & -R^2 CS - 2R \\
  RCS^2 + 2CS & -RCS - 1
\end{bmatrix}
\begin{bmatrix}
  e_{02} \\
  \eta_{02}
\end{bmatrix}
\]

With \( \eta_{02} = 0 \)

\[
\frac{e_{02}}{e_{r1}} = \frac{1}{(RCS + 1)^2 + RCS}
\]

Using Impedances:

\[
\frac{e_{01}}{e_{r1}} = \mathcal{W}_u(s) \left[ \frac{1}{1 + \frac{Z_{g_01}(s)}{Z_{g21}(s)}} \right]
\]

\[
\mathcal{W}_u(s) = \frac{e_{01}(s)}{e_{r1}(s)} \bigg|_{f_{01}(s) = 0} = \frac{1}{RCS + 1}
\]

\[
Z_{g01}(s) = \frac{e_{01}(s)}{f_{01}(s)} \bigg|_{e_{r1}(s) = 0} = \frac{R}{RCS + 1}
\]
\[
\frac{2 \cdot \text{ou}_2(s)}{\text{f}_{12}(s)} = \left. \frac{\text{e}_{12}(s)}{\text{f}_{12}(s)} \right|_{\text{f}_{02}(s)=0} = \frac{\text{RCS}+1}{\text{Cs}}
\]

Therefore

\[
\frac{\text{e}_{01}}{\text{e}_{11}} = \frac{1}{\text{RCs}+1} \left[ \frac{1}{1 + \frac{\text{RCs}}{(\text{RCs}+1)^2}} \right]
\]

This is the loaded transfer function for the first RC Circuit. To get the overall transfer function we multiply this by the ideal transfer function of the second RC Circuit since it is unloaded.

\[
\frac{\text{e}_{02}}{\text{e}_{11}} = \frac{1}{\text{RCs}+1} \left[ \frac{1}{1 + \frac{\text{RCs}}{(\text{RCs}+1)^2}} \right] \frac{1}{\text{RCs}+1}
\]

\[
= \frac{1}{(\text{RCs}+1)^2 + \text{RCs}} \quad \text{Same as found by direct analysis.}
\]
Mechanical System

\[
\begin{bmatrix}
   f_{x1} \\
   x_{x1}
\end{bmatrix} = 
\begin{bmatrix}
   1 & m_1s^2 + b_1s \\
   0 & 1
\end{bmatrix} 
\begin{bmatrix}
   f_{o1} \\
   x_{o1}
\end{bmatrix}
\]

\[
\begin{bmatrix}
   f_{x2} \\
   x_{x2}
\end{bmatrix} = 
\begin{bmatrix}
   1 & K_2 \\
   \frac{1}{b_2s} & \frac{1}{b_2s + K_2}
\end{bmatrix} 
\begin{bmatrix}
   f_{o2} \\
   x_{o2}
\end{bmatrix}
\]

Desired Transfer Function: \( \frac{x_{o2}}{f_{x1}} \)

\( x_{o1} = x_{x2} \) when connected

\( f_{o1} = f_{x2} \)

Assume \( f_{o2} = 0 \), i.e., there is no system connected to the output of system 2.

\[
\frac{x_{o2}}{f_{x1}} = \left[ \frac{x_{o1}}{f_{x1}} \right]_{\text{loaded}} \left[ \frac{x_{o2}}{x_{x2}} \right]_{\text{Unloaded}}
\]
\[ \chi_{01} = \chi_{01} (f_{x1}, f_{01}) \]
\[ = \frac{\chi_{01}}{f_{x1}} \bigg|_{f_{x1}=0} f_{01} + \frac{\chi_{01}}{f_{01}} \bigg|_{f_{x1}=0} f_{01} \]
\[ f_{01} = f_{x2} \] \{ \text{systems are connected,} \}
\[ \chi_{01} = \chi_{x2} \]

\[ f_{x2} = f_{x2} (f_{02}, \chi_{x2}) \]
\[ = \frac{f_{x2}}{f_{02}} \bigg|_{\chi_{x2}=0} f_{02} + \frac{f_{x2}}{\chi_{x2}} \bigg|_{f_{02}=0} \chi_{x2} \]
\[ = \frac{f_{x2}}{\chi_{x2}} \bigg|_{f_{02}=0} \chi_{x2} \quad \text{since } f_{02} = 0 \]
\[ = \frac{f_{x2}}{\chi_{x2}} \bigg|_{f_{02}=0} \chi_{01} \quad \text{since } \chi_{01} = \chi_{x2} \]
\[ = f_{01} \quad \text{since } f_{x2} = f_{01} \]
Therefore

\[ \chi_{01} = \frac{\chi_{01}}{f_{x1}} \left. f_{x1} \right|_{f_{01}=0} + \frac{\chi_{01}}{f_{01}} \left. \frac{f_{x2}}{\chi_{x2}} \right|_{f_{x1}=0} \left. f_{x2} \right|_{f_{02}=0} \chi_{01} \]

\[ \chi_{01} = \frac{\chi_{01}}{f_{x1}} \left. f_{x1} \right|_{f_{01}=0} + \frac{\chi_{12}}{f_{12}} \left. f_{12} \right|_{f_{02}=0} \chi_{01} \]

Solve for \( \frac{\chi_{01}}{f_{x1}} \):

\[ \frac{\chi_{01}}{f_{x1}} = \frac{\chi_{01}}{f_{x1}} \left. f_{x1} \right|_{f_{01}=0} \left[ \frac{1}{1 - \frac{\chi_{01}}{f_{01}} \left. f_{x1} \right|_{f_{01}=0}} \right] \]

Apply this to systems 1 and 2:

\[
\begin{bmatrix}
  f_{x1} \\
  \chi_{x1}
\end{bmatrix} = \begin{bmatrix}
  1 & m_1 s^2 + b_1 s \\
  0 & 1
\end{bmatrix} \begin{bmatrix}
  f_{01} \\
  \chi_{01}
\end{bmatrix}
\]

\[
\begin{bmatrix}
  f_{x2} \\
  \chi_{x2}
\end{bmatrix} = \begin{bmatrix}
  1 & \frac{K_2}{b_2 s^2 + K_2} \\
  \frac{1}{b_2 s} & \frac{K_2}{b_2 s}
\end{bmatrix} \begin{bmatrix}
  f_{02} \\
  \chi_{02}
\end{bmatrix}
\]
\[
\frac{X_{01}}{F_{x1}} \bigg|_{f_{01}=0} = \frac{1}{m_1 s^2 + b_1 s}
\]

\[
\frac{X_{01}}{F_{x1}} \bigg|_{f_{x1}=0} = \frac{-1}{m_1 s^2 + b_1 s}
\]

\[
\frac{X_{12}}{F_{x2}} \bigg|_{f_{02}=0} = \frac{b_2 s + K_2}{K_2 b_2 s}
\]

Combine:

\[
\frac{X_{01}}{F_{x1}} = \frac{1}{m_1 s^2 + b_1 s} \left[ \frac{1}{1 - \frac{-1/m_1 s^2 + b_1 s}{b_2 s + K_2/K_2 b_2 s}} \right]
\]

\[
= \frac{b_2 s + K_2}{(m_1 s^2 + b_1 s)(b_2 s + K_2) + b_2 K_2 s}
\]

= Loaded Transfer Function for System #1
\[
\frac{X_{o2}}{X_{a2}} \bigg|_{f_{o2}=0} = \frac{b_2 s}{b_2 s + K_2}
\]

[Unloaded Transfer Function for System \#2]

Result:

\[
\frac{X_{o2}}{f_{a1}} = \frac{b_2 s + K_2}{(m_1 s^2 + b_1 s)(b_2 s + K_2) + b_2 K_2 s} \cdot \frac{b_2 s}{b_2 s + K_2}
\]

\[
= \frac{b_2}{(b_1 + b_2) K_2}
\]

\[
\frac{m_1 b_2}{(b_1 + b_2) K_2} s^2 + \frac{m_1 K_2 + b_1 b_2}{(b_1 + b_2) K_2} s + 1
\]

Same as obtained previously.
Hydraulic System

\[
\begin{bmatrix}
  h_{x1} \\
  q_{x1}
\end{bmatrix} =
\begin{bmatrix}
  1 & R_1 \\
  A_1 s & A_1 R_1 s + 1
\end{bmatrix}
\begin{bmatrix}
  h_{01} \\
  q_{01}
\end{bmatrix}
\]

\[
\begin{bmatrix}
  h_{x2} \\
  q_{x2}
\end{bmatrix} =
\begin{bmatrix}
  1 & R_2 \\
  A_2 s & A_2 R_2 s + 1
\end{bmatrix}
\begin{bmatrix}
  h_{02} \\
  q_{02}
\end{bmatrix}
\]

\[
q_{01} = q_{x2} \quad \text{when connected} \\
q_{02} = 0 \Rightarrow \text{unloaded}
\]

Desired Transfer Function: \( \frac{q_{02}}{q_{x1}} \)

From previous analysis:

\[
\left. \frac{q_{02}}{q_{x1}} \right|_{h_{02} = 0} = \frac{1}{(A_1 R_1 s + 1)(A_2 R_2 s + 1) + A_1 R_1}
\]

Now use Impedance.
\[ q_{01} = q_{01} \left( q_{11}, h_{01} \right) \]
\[ = \frac{q_{01}}{q_{11}} \bigg|_{h_{01} = 0} + \frac{q_{01}}{h_{01}} \bigg|_{q_{11} = 0} \]

\[ q_{12} = q_{12} \left( h_{12}, h_{02} \right) \]
\[ = \frac{q_{12}}{h_{12}} \bigg|_{h_{02} = 0} + \frac{q_{12}}{h_{02}} \bigg|_{h_{12} = 0} \]

But: \[ h_{02} = 0 \]
\[ q_{12} = q_{01} \]
\[ h_{01} = h_{12} \]

\[ q_{01} = \frac{q_{01}}{q_{11}} \bigg|_{h_{01} = 0} + \frac{q_{01}}{h_{01}} \bigg|_{q_{11} = 0} \]
Solve for \( \frac{g_{01}}{g_{11}} \):

\[
\begin{align*}
\frac{g_{01}}{g_{11}} &= \frac{g_{01}}{g_{11}} \left| \begin{array}{c}
\frac{1}{h_{01} = 0} \\
\frac{1}{h_{12} = 0}
\end{array} \right. \\
&= \frac{1}{A_1 R_1 s + 1} \left| \begin{array}{c}
\frac{g_{01}}{h_{01} = 0} \\
\frac{-A_1 s}{A_1 R_1 s + 1}
\end{array} \right. \\
&= \frac{A_2 R_2 s + 1}{R_2} \left| \begin{array}{c}
\frac{g_{12}}{h_{12} = 0} \\
\frac{h_{02} = 0}
\end{array} \right. 
\end{align*}
\]
\[
\frac{g_{01}}{g_{n1}} = \frac{1}{A_1 R_s + 1} \left[ \frac{1}{1 - \frac{A_1 R_s}{A_2 R_s + 1}} \right]
\]

\[
\frac{g_{02}}{g_{n1}} = \frac{g_{01}}{g_{n1}} \quad \text{Loaded}
\]

\[
\frac{g_{02}}{g_{n2}} = \frac{g_{02}}{g_{n2}} \quad \text{Unloaded}
\]

Note that \( g_{01} = g_{n2} \)

\[
\frac{g_{02}}{g_{n2}} \bigg|_{h_0 = 0} = \frac{1}{A_2 R_s + 1}
\]

Result:

\[
\frac{g_{02}}{g_{n1}} = \frac{1}{A_1 R_s + 1} \left[ \frac{1}{1 + \frac{A_1 R_s}{(A_1 R_s + 1)(A_2 R_s + 1)}} \right]
\]

Same as previously determined.
III. Practical Application

- In equipment involving diverse technologies, major subsystems may be manufactured by different contractors at remote locations, with subsystems being brought together at a single final assembly point only after each has been individually completed.

- Experimental studies of the complete system cannot be performed until final assembly. Discovery of design faults at this late stage can cause severe economic and scheduling problems.

- Capability for experimental testing of each subsystem at the respective manufacturer's facility and proper coupling of these results to predict behavior of the assembled system can be a valuable tool.
• Consider the following physical situation:

![Diagram showing a motor, structure, and electronics package with forces and velocities indicated.]

- Pure Translation (Unidirectional motions assumed for simplicity.)

- Motor produces vibration-exciting dynamic forces $f_x1$ at location $A$.

- Electronic package at location $B$ can withstand only limited vibration.

- Determine the force $f_x2$ that will be applied to the electronics package when it has been connected.

$f \equiv$ force  
$v \equiv$ velocity
• It is preferable to run separate vibration tests on each subsystem and then calculate from these measurements what the force will be.

• Assume that the frequency spectrum of the input force \( f_{x1} \) is known from theory or experiment.

Problem: Find \( \frac{f_{x2}}{f_{x1}}(s) \) for the loaded condition

\[ f_{x2} = f_{x1} \] when the subsystems are joined.

\[ f_{o2} = 0 \] since the electronics package is allowed to vibrate freely.

\[ V_{01} = V_{12} \] when the subsystems are joined.
\[ f_{01} = f_{01} (f_{\alpha_1}, v_{01}) \]
\[ f_{01} = \frac{f_{01}}{f_{\alpha_1}} \bigg|_{v_{01}=0} f_{\alpha_1} + \frac{f_{01}}{v_{01}} \bigg|_{f_{\alpha_1}=0} v_{01} \]

\[ f_{\alpha_2} = f_{\alpha_2} (v_{\alpha_2}, f_{02}) \]
\[ f_{\alpha_2} = \frac{f_{\alpha_2}}{v_{\alpha_2}} \bigg|_{f_{02}=0} v_{\alpha_2} + \frac{f_{\alpha_2}}{f_{02}} \bigg|_{v_{\alpha_2}=0} f_{02} \]

Combine:
\[ f_{01} = \frac{f_{01}}{f_{\alpha_1}} \bigg|_{v_{01}=0} f_{\alpha_1} + \frac{f_{01}}{v_{01}} \bigg|_{f_{\alpha_1}=0} \begin{bmatrix} -f_{\alpha_2} \\ f_{\alpha_2} \\ v_{\alpha_2} \\ f_{02}=0 \end{bmatrix} \]

But \[ f_{\alpha_2} = f_{01} \]
\[
\frac{f_{01}}{f_{a1}} = \frac{f_{01}}{f_{a1}} \left[ \frac{1}{1 + \frac{f_{01}}{v_{01}}} \right]
\]

Experimental Measurements needed to predict coupled-system response.