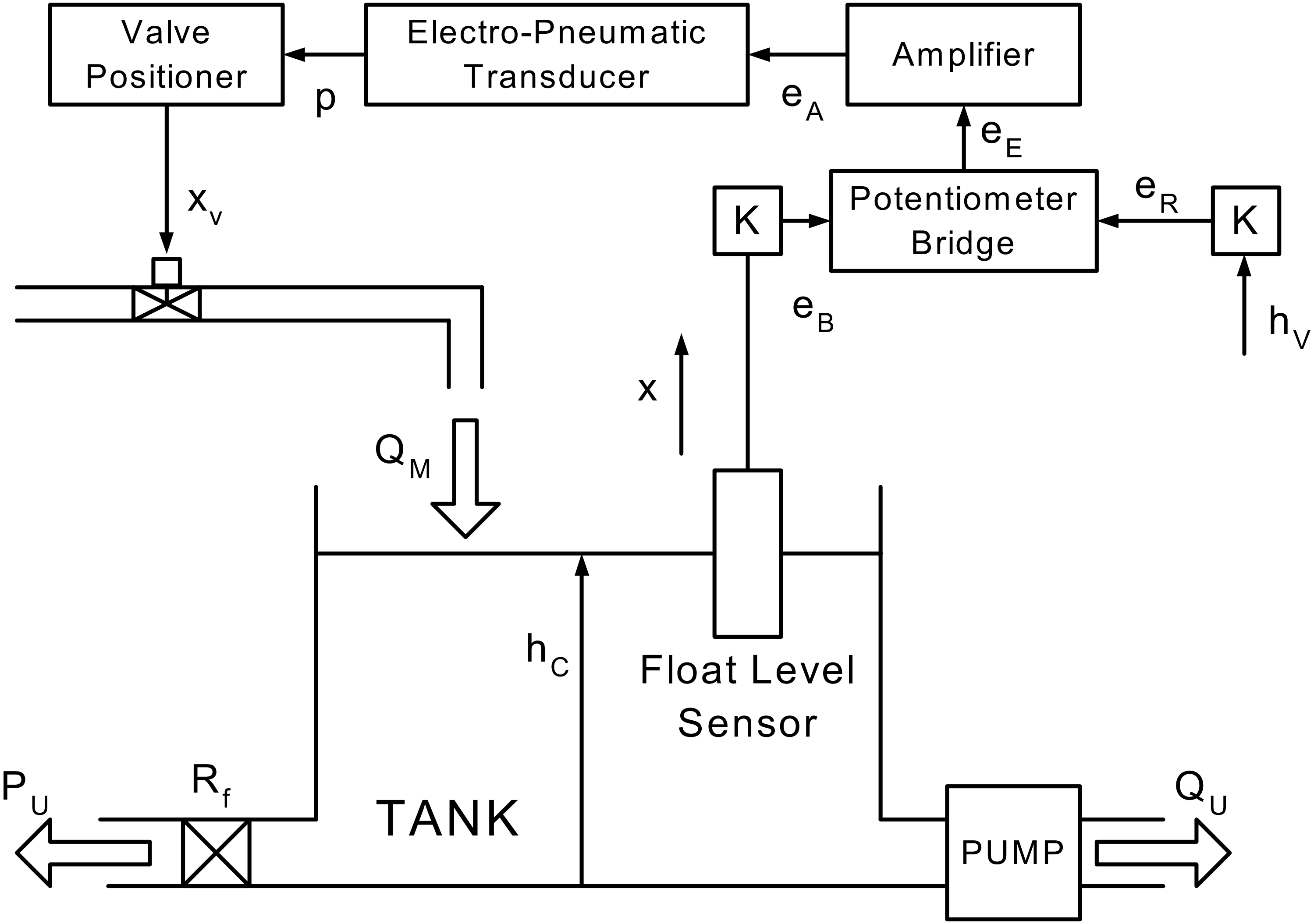
Control of a Liquid-Level Process



- <u>Objective</u>
  - Maintain tank level h_C at the desired level h_V in the face of disturbances pressure $P_U(t)$ (psig) and volume flow rate $Q_U(t)$ (ft^3/sec). R_f is a linearized flow resistance with units psi/(ft^3/sec).
- <u>Equilibrium Operating Point</u>
  - All variables are steady
  - Inflow Q_M exactly matches the two outflows
  - $h_C = h_V$ and $e_E = 0$
    - When $e_E = 0$, Q_M can be nonzero since the electropneumatic transducer has a zero adjustment and the valve positioner has a zero adjustment, e.g., p = 9 psig and the valve opening corresponds to equilibrium flow Q_M.
    - We will deal with small perturbations in all variables away from the initial steady state.

- Assumptions and Equations of Motion
- Tank Process Dynamics
  - Density of fluid ρ is constant.

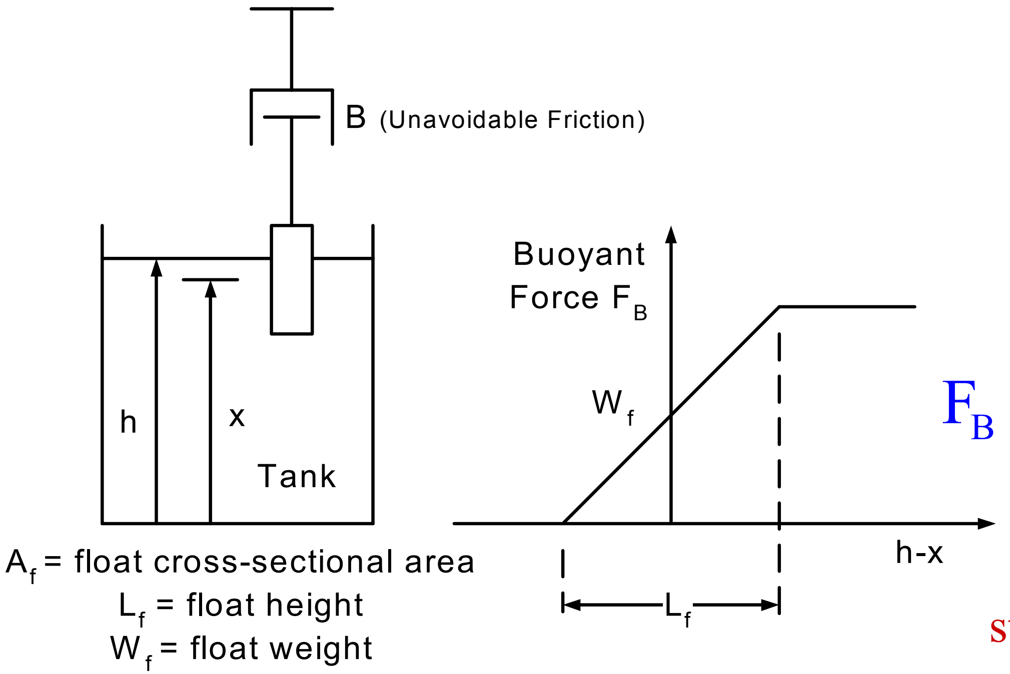
$$Q_M - Q_U - \frac{\rho g h_C - P_U}{R_f} = A_T \frac{dh_C}{dt}$$

$$\left(\tau_p s + 1\right) h_C = \frac{R_f}{\rho g} Q_M + \frac{1}{\rho g} P_U - \frac{R_f}{\rho g} Q_U$$

Conservation of Mass

$$\tau_p = \frac{A_T R_f}{\rho g}$$ process time constant

- <u>Float Level Sensor</u>
  - Assume a zero-order dynamic model, i.e., the dynamics are negligible relative to the process time constant τ_p since the cross-sectional area of the tank is assumed large.
  - Consider the actual dynamics to justify this assumption:



$$F_B = W_f + A_f \rho g (h - x)$$

steady condition: h = x

– Equation of Motion

$$F_B - W_f - B\frac{dx}{dt} = M_f \frac{d^2x}{dt^2}$$

$$M_f \frac{d^2x}{dt^2} + B\frac{dx}{dt} = A_f \rho g \left(h - x\right)$$

$$M_f \frac{d^2x}{dt^2} + B\frac{dx}{dt} + A_f \rho g x = A_f \rho g h$$

$$\left[\frac{D^2}{\omega_n^2} + \frac{2\zeta D}{\omega_n} + 1\right] x = Kh$$

$$\omega_n = \sqrt{\frac{A_f \rho g}{M_f}}$$

$$D \equiv \frac{d}{dt}$$

$$D^2 \equiv \frac{d^2}{dt^2}$$

Differential Operator

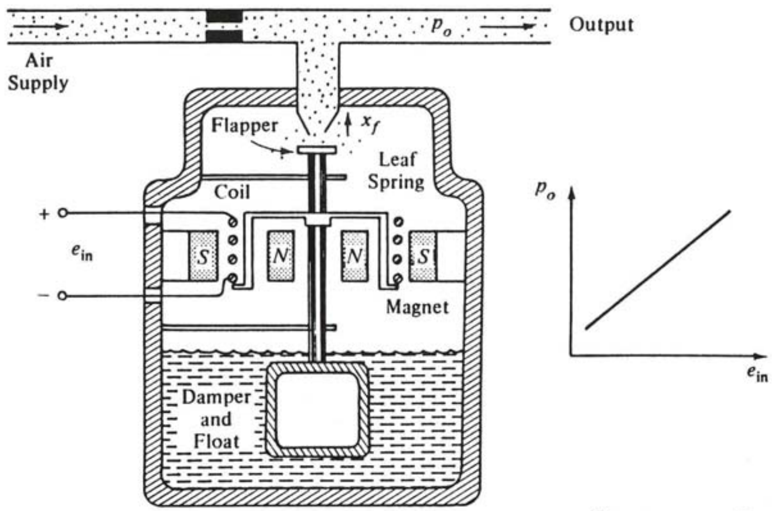
$$\zeta = \frac{B}{2\sqrt{A_f M_f \rho g}}$$

$$K = 1.0$$

- To measure rapid changes in h accurately, ω_n must be sufficiently large. The specific weight of the fluid (ρg) is not a design variable, so strive for large values of A_f/M_f (i.e., hollow floats).
- In our case, the tank has a large diameter and if the inflow and outflow rates are modest, h cannot change rapidly and so a zero-order model is justified.

- Potentiometer Bridge and Electronic Amplifier
  - Obviously these two components are fast enough to be treated as zero order in this system.

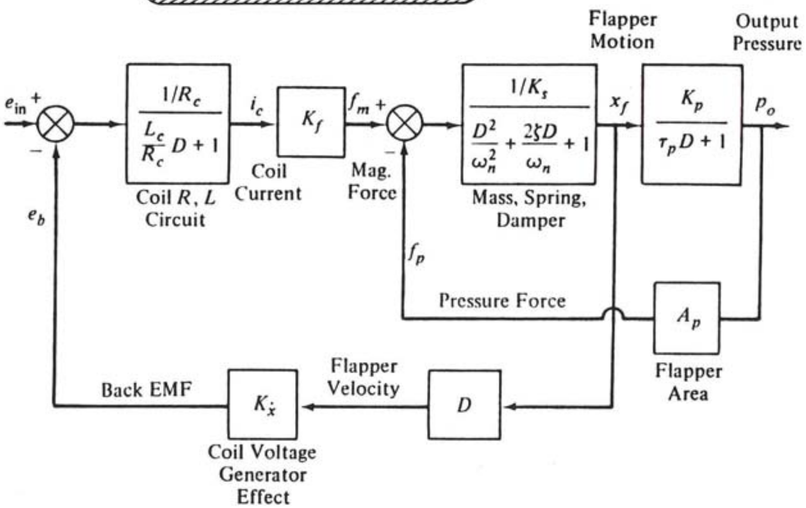
• Electropneumatic Transducer



This device produces a pneumatic output signal closely proportional (± 5% nonlinearity) to an electrical input (± 5V and 3-15 psig).

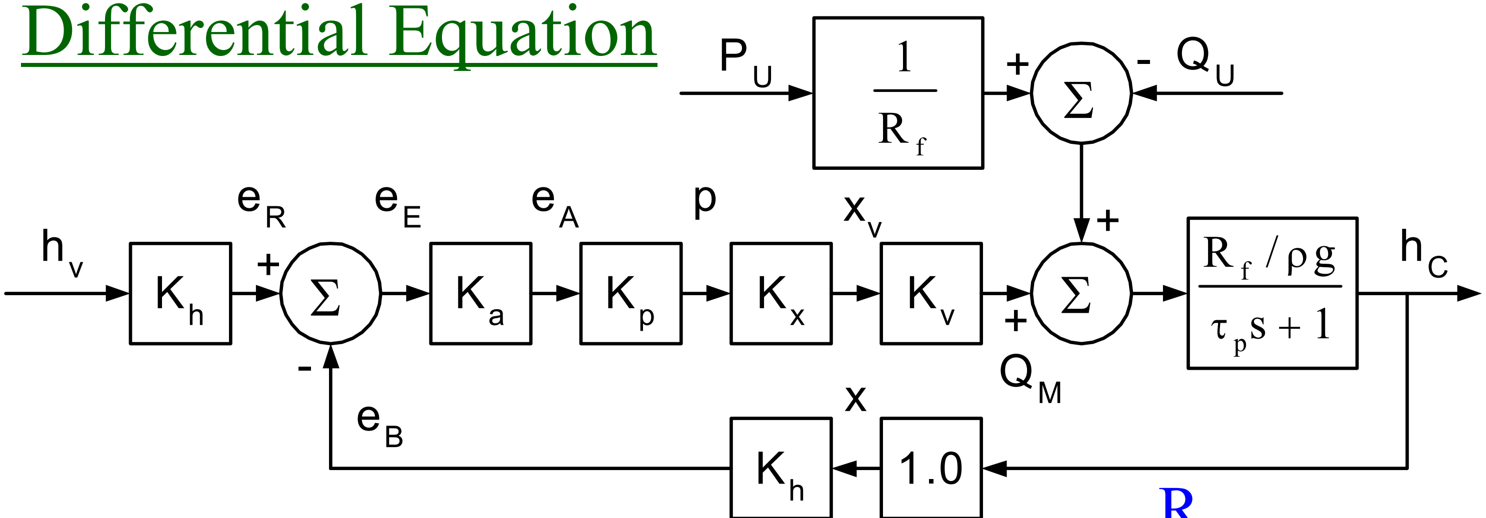
We are concerned with overall dynamics from e_A to p. The block diagram shows a 4th-order closed-loop differential equation.

However, experimental frequency response tests show typically a flat amplitude ratio out to about 5 Hz. This response is very fast relative to τ_p so we model the electropneumatic transducer as zero order.



- Pneumatic Valve Positioner
  - We are only interested in the overall dynamics relating x_v to p. These are again quite fast relative to τ_p, so we model the component as zero order.
  - The valve positioner allows one to "characterize" the static calibration curve between p and x_v and thus obtain desired linear or nonlinear relationships between p and manipulated flowrate Q_M.
- Relation between Q_M and x_v
  - This relationship is assumed to be statically linear and dynamically instantaneous and thus a zero-order model.
  - Although the dynamic response of Q_M to x_v is not instantaneous due to fluid inertia and compliance, the response is much faster than the tank-filling dynamics.

- <u>Closed-Loop System Block Diagram and Differential Equation</u>



$$\left[\left(K_h h_V - K_h h_C\right)K_a K_p K_x K_v + \frac{1}{R_f}P_U - Q_U\right]\frac{\frac{R_f}{\rho g}}{\tau_p s+1} = h_C$$

$$\left(\tau_s s+1\right)h_C = \frac{K}{K+1}h_V + \frac{1}{\rho g\left(K+1\right)}P_U - \frac{R_f}{\rho g\left(K+1\right)}Q_U$$

$$\tau_s = \frac{\tau_p}{K+1}$$

Closed-Loop System Time Constant

$$K = \frac{1}{\rho g}\left(K_h K_a K_p K_x K_v R_f\right)$$

System Loop Gain

- <u>Speed of Response</u>
  - Response for a step input in h_v (hold perturbations P_U and Q_U at zero)

$$h_C = \frac{K}{K+1} h_{V_s} \left(1 - e^{\frac{-t}{\tau_s}} \right)$$

  - Response for a step input in disturbances P_U and Q_U (hold $h_V = 0$)

$$h_C = \frac{1}{\rho g (K+1)} P_{U_s} \left(1 - e^{\frac{-t}{\tau_s}} \right) \qquad h_C = \frac{-R_f}{\rho g (K+1)} Q_{U_s} \left(1 - e^{\frac{-t}{\tau_s}} \right)$$

  - Increasing loop gain K increases the speed of response

$$\tau_s = \frac{\tau_p}{K+1} = \text{closed-loop system time constant}$$

- Steady-State Errors
  - A procedure generally useful for all types of systems and inputs is to rewrite the closed-loop system differential equation with system error (V-C), rather than the controlled variable C, as the unknown.
  - In this case we have:

$$\left(\tau_s s+1\right)h_C=\frac{K}{K+1}h_V+\frac{1}{\rho g\left(K+1\right)}P_U-\frac{R_f}{\rho g\left(K+1\right)}Q_U$$

$$h_E=h_V-h_C$$

$$\left(\tau_s s+1\right)\left(h_V-h_E\right)=\frac{K}{K+1}h_V+\frac{1}{\rho g\left(K+1\right)}P_U-\frac{R_f}{\rho g\left(K+1\right)}Q_U$$

$$\left(\tau_s s+1\right)h_E=\left(\tau_s s+\frac{1}{K+1}\right)h_V-\frac{1}{\rho g\left(K+1\right)}P_U+\frac{R_f}{\rho g\left(K+1\right)}Q_U$$

– For any chosen commands or disturbances, the steady-state error will just be the particular solution of the equation:

$$\left(\tau_s s+1\right)h_E=\left(\tau_s s+\frac{1}{K+1}\right)h_V-\frac{1}{\rho g\left(K+1\right)}P_U+\frac{R_f}{\rho g\left(K+1\right)}Q_U$$

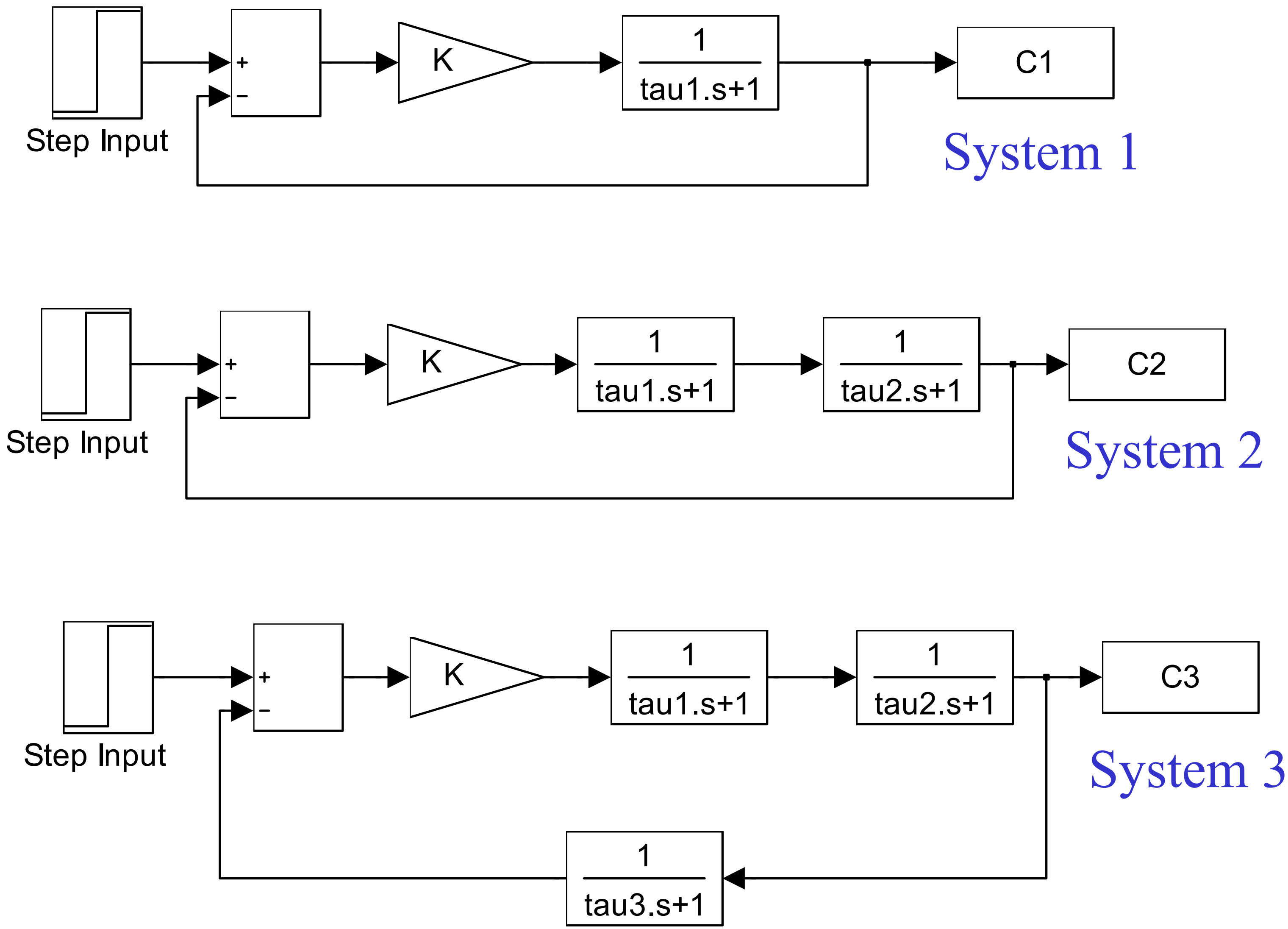
– We see that the steady-state error is improved if we increase the loop gain K.
– For any initial equilibrium condition we can "trim" the system for zero error but subsequent steady commands and/or disturbances must cause steady-state errors.
– Ramp inputs would cause steady-state errors that increase linearly with time, the rate of increase being proportional to ramp slope and inversely proportional to K+1.

- Stability
  - All aspects of system behavior are improved by increasing loop gain – up to a point! – instability may result, but our present model gives no warning of this. Why?
  - We neglected dynamics in some components and a general rule is:

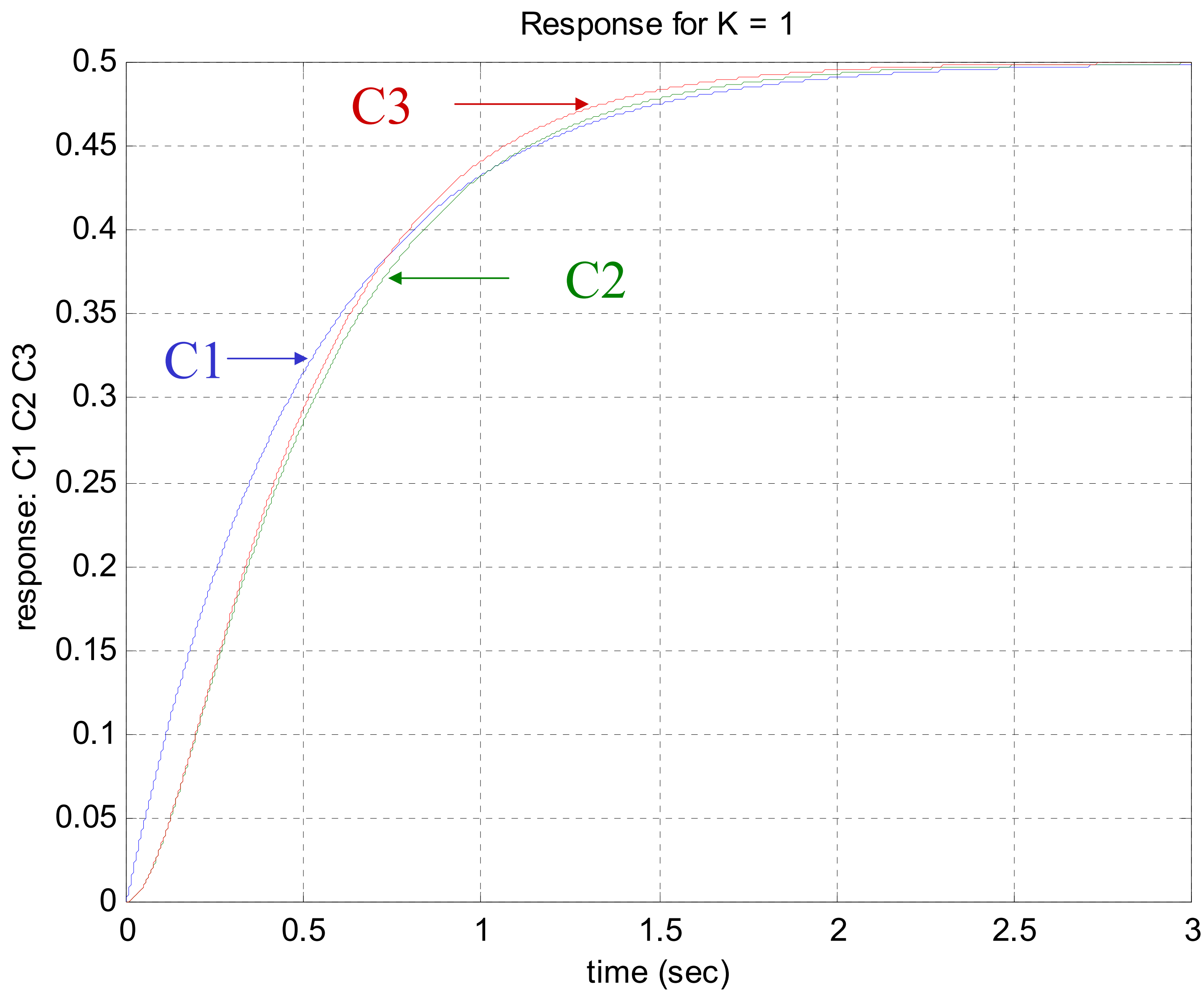
    If we want to make valid stability predictions we must include enough dynamics in our system so that the closed-loop system differential equation is at least third order. The one exception is systems with dead times where instability can occur even when dynamics are zero, first, or second order.
  - Is our model then useless?

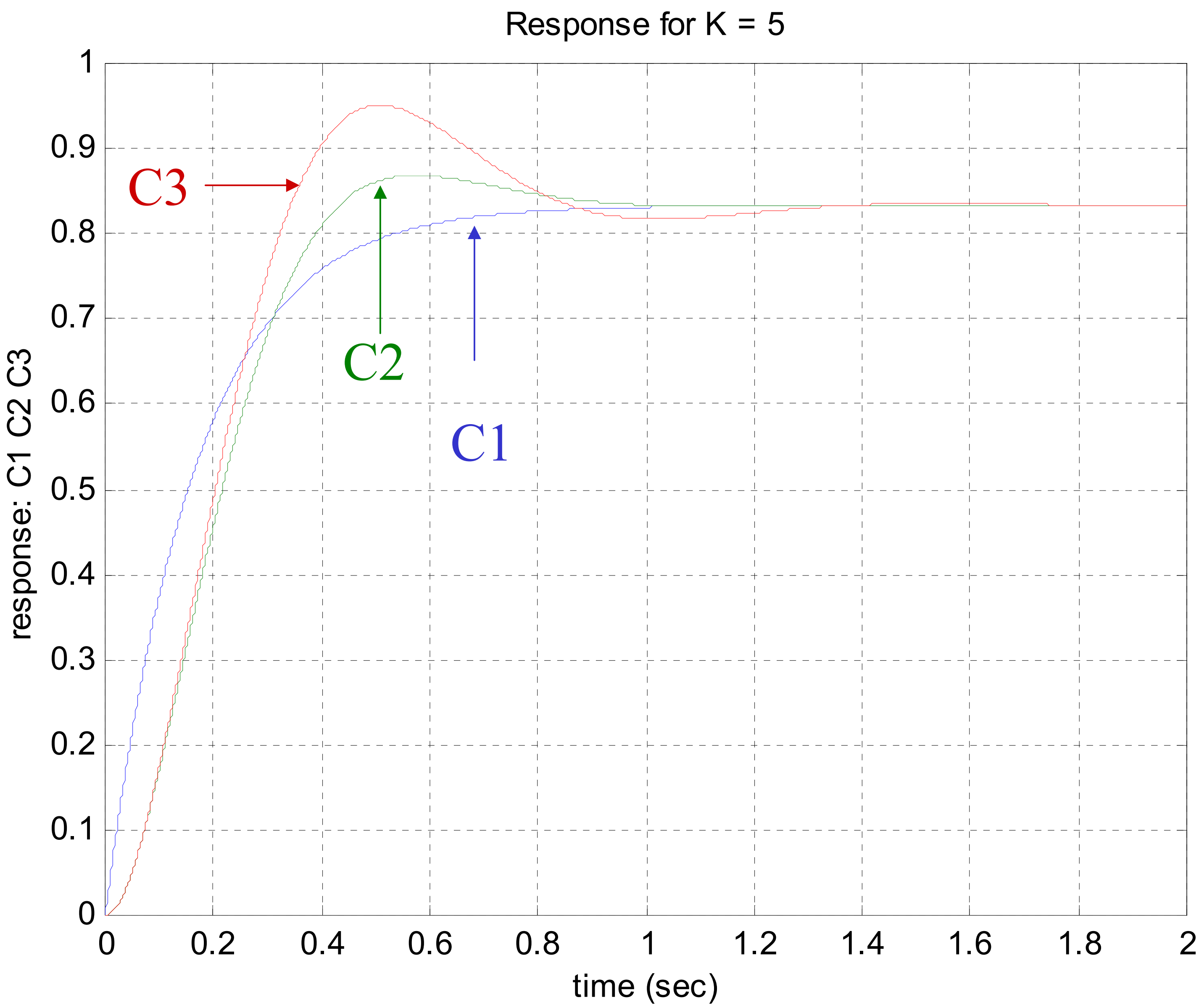
– No! It does correctly predict system behavior as long as the loop gain is not made "too large." As K is increased, the closed-loop system response gets faster and faster. At some point, the neglected dynamics are no longer negligible and the model becomes inaccurate.
– We neglected dynamics relative to τ_p, but in the closed-loop system, response speed is determined by τ_s.

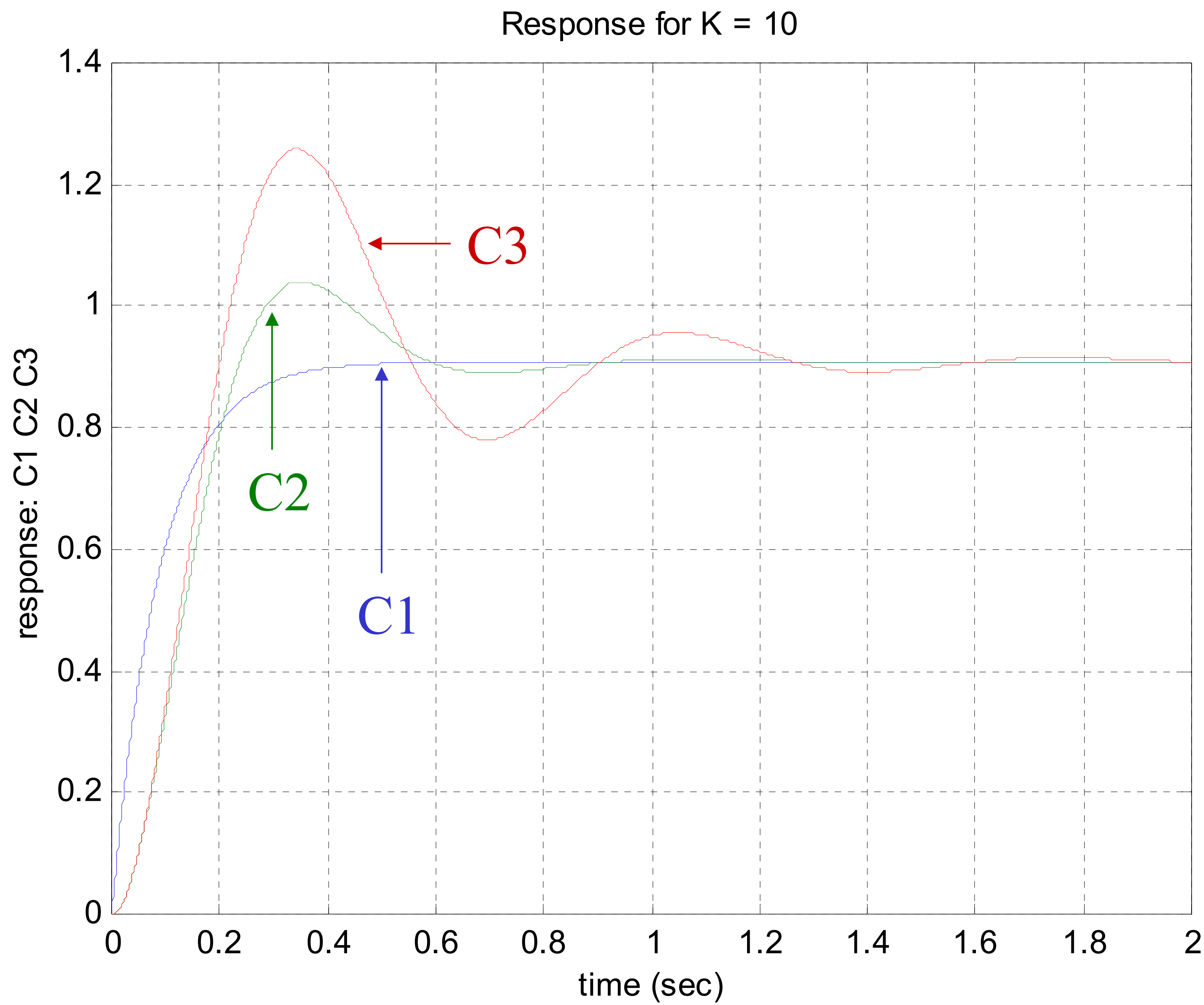
- Exercise:
  – Compare the responses of the following 3 systems for K = 1, 5, 10 with $\tau_1 = 1.0$, $\tau_2 = 0.1$, and $\tau_3 = 0.05$. The input is a unit step.
  – Examine speed of response, steady-state error, and stability predictions.

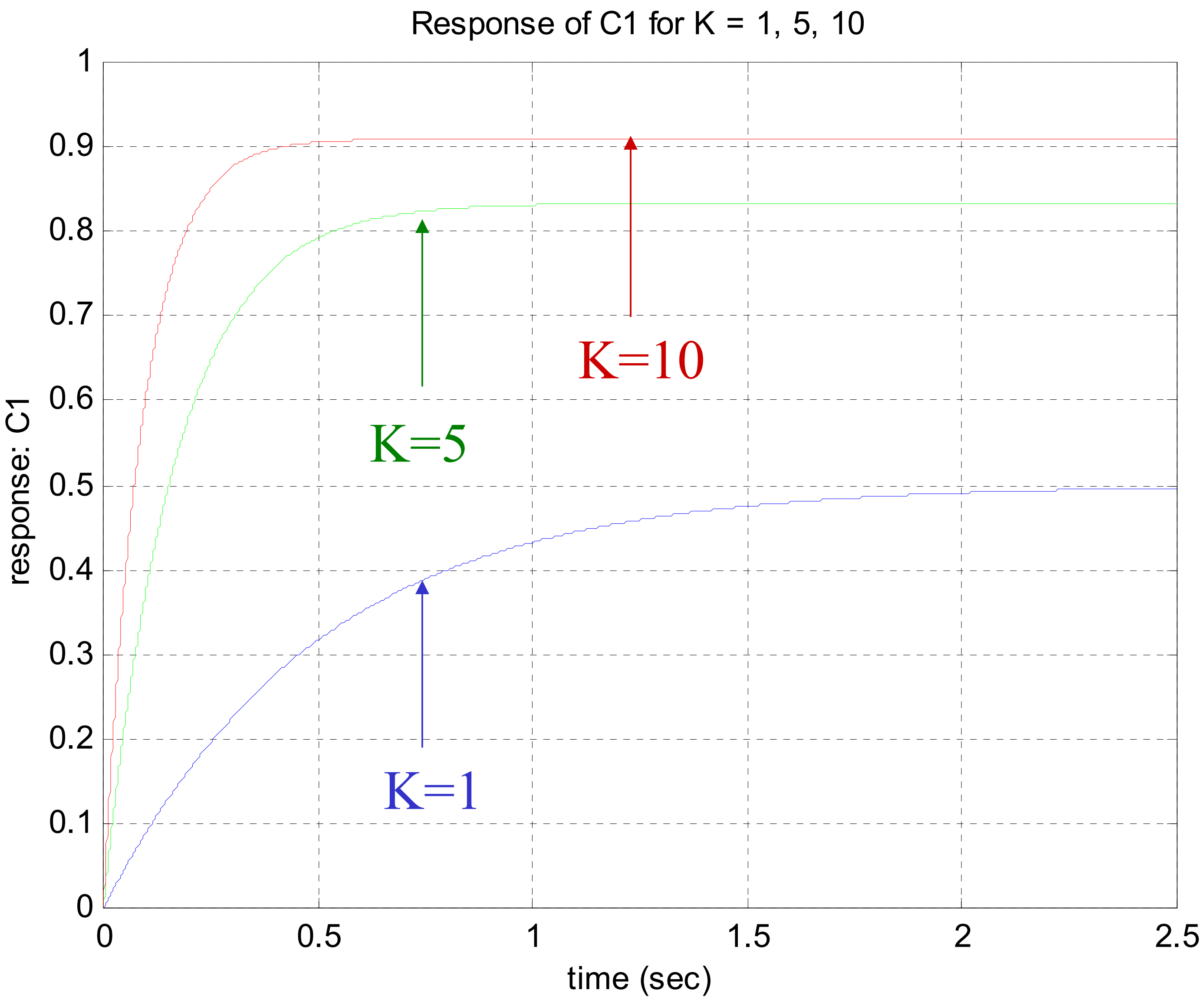


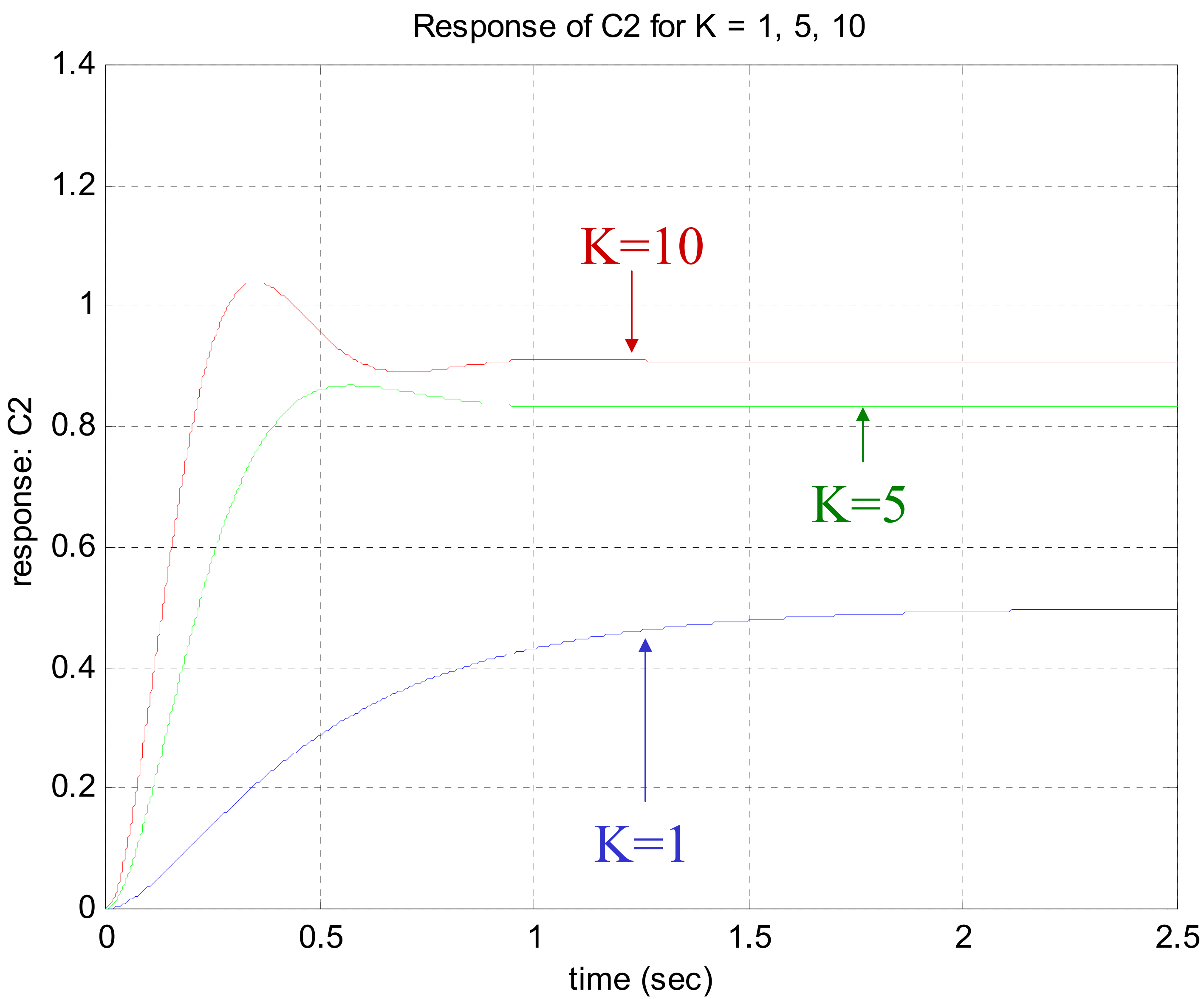
MatLab / Simulink Diagram

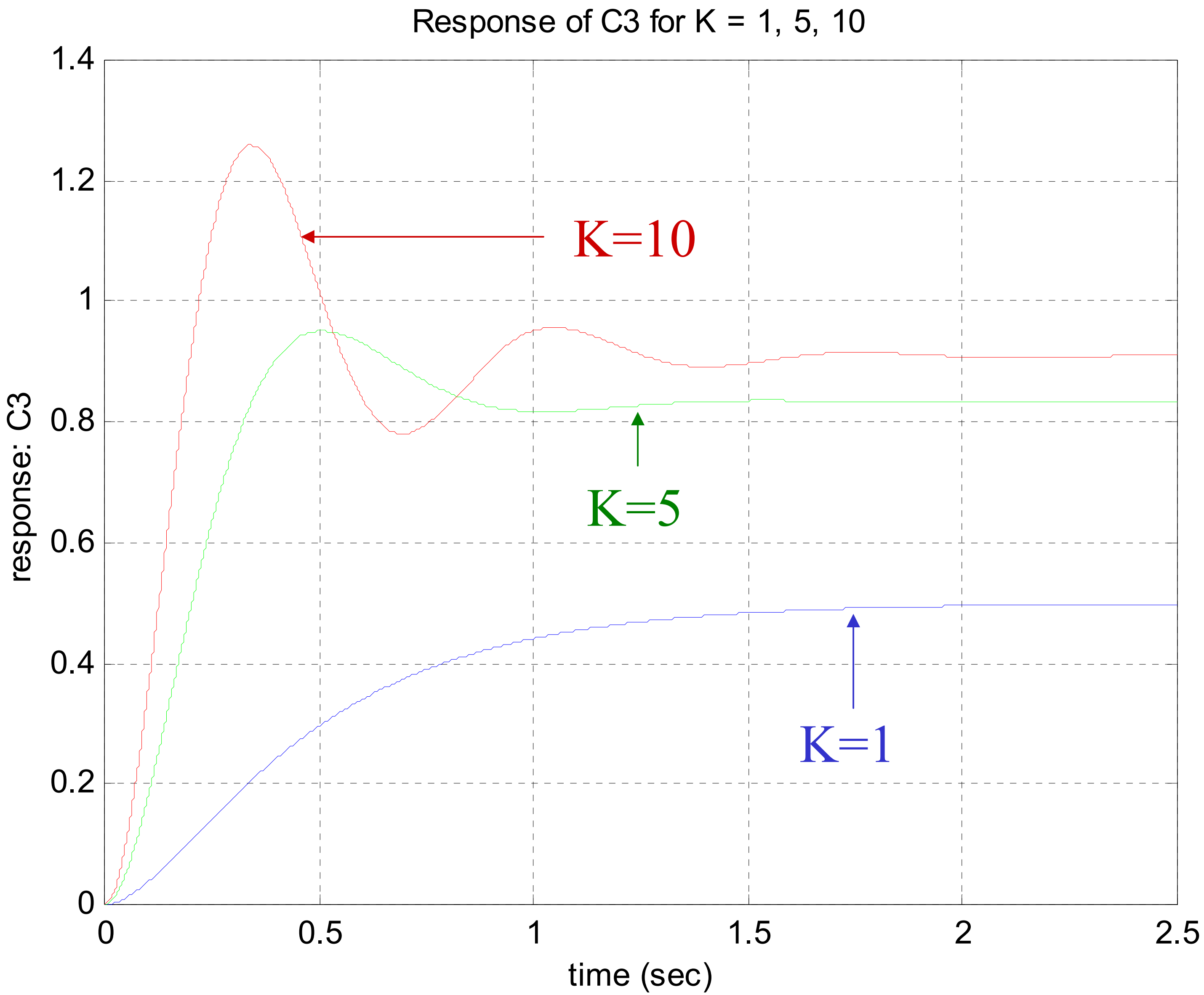

Response for K = 1
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
0.5
1
1.5
2
2.5
3
time (sec)
response: C1 C2 C3
C1
C2
C3


Response for K = 5
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
C3
C2
C1
response: C1 C2 C3
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
time (sec)


Response for K = 10
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
time (sec)
response: C1 C2 C3
C3
C2
C1


Response of C1 for K = 1, 5, 10
response: C1
time (sec)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.5
1
1.5
2
2.5
K=10
K=5
K=1


Response of C2 for K = 1, 5, 10
1.4
1.2
1
0.8
0.6
0.4
0.2
0
response: C2
0
0.5
1
1.5
2
2.5
time (sec)
K=10
K=5
K=1


Response of C3 for K = 1, 5, 10
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
response: C3
time (sec)
K=10
K=5
K=1

- Note that as loop gain K is increased, the speed of response is increased and the steady-state error is reduced.
- For what value of loop gain K will any of these systems go unstable?
- Let's look at the closed-loop system transfer functions and characteristic equations:

Transfer Functions
(The characteristic equation is obtained by setting the denominator polynomial equal to zero.)

$$\frac{C1}{V} = \frac{K}{\tau_1 s + 1 + K}$$

$$\frac{C2}{V} = \frac{K}{\tau_1 \tau_2 s^2 + (\tau_1 + \tau_2) s + 1 + K}$$

$$\frac{C3}{V} = \frac{K(\tau_3 + 1)}{\tau_1 \tau_2 \tau_3 s^3 + (\tau_1 \tau_2 + \tau_2 \tau_3 + \tau_1 \tau_3) s^2 + (\tau_1 + \tau_2 + \tau_3) s + 1 + K}$$

$$\frac{C1}{V} = \frac{K}{s + 1 + K}$$

$$\frac{C2}{V} = \frac{K}{0.1s^2 + 1.1s + 1 + K}$$

$$\frac{C3}{V} = \frac{K(0.05s + 1)}{0.005s^3 + 0.155s^2 + 1.15s + 1 + K}$$

– The only system which will go unstable as the loop gain K is increased is the third system; its characteristic equation is third order. The first two systems will continue to show improved speed of response and reduction of steady-state error without any hint of instability!

- Let's apply the three methods of determining closed-loop system stability – Routh, Nyquist, and Root-Locus - to the third system and determine the value of K for which this system becomes marginally stable.
- Routh Stability Criterion
  - Closed-Loop System Characteristic Equation

$$0.005s^3 + 0.155s^2 + 1.15s + 1 + K = 0$$

  - Routh Array

0.005	1.15
0.155	$1+K$
$\dfrac{(0.155)(1.15)-(1+K)(0.005)}{0.155}$	0
$1+K$	0

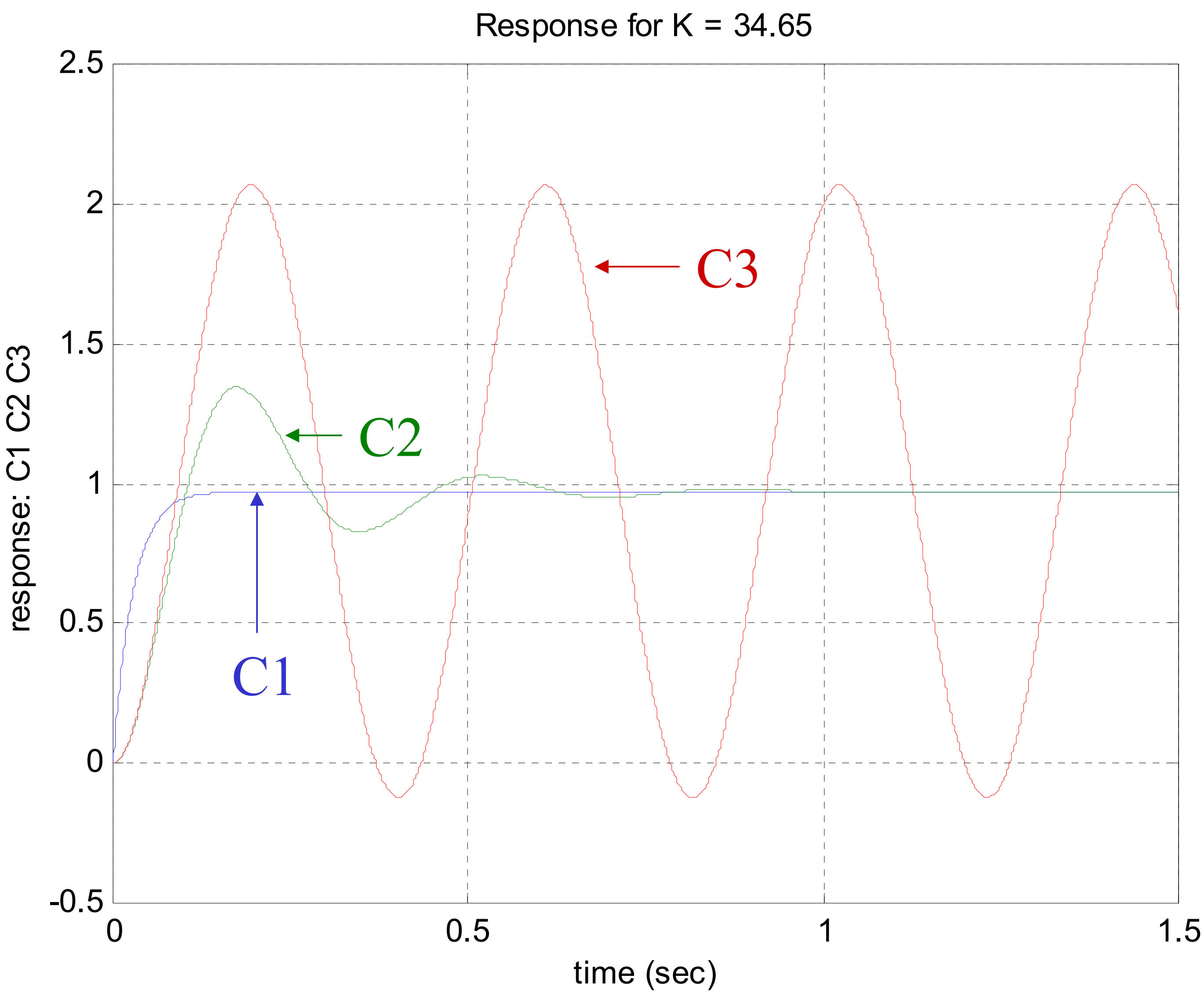
- For stability we see that:

$$(0.155)(1.15) - (1 + K)(0.005) > 0$$
$$1 + K > 0$$

- This leads to the result that for absolute stability:

$$-1 < K < 34.65$$

- A simulation with the loop gain set to K = 34.65 should verify this result. The value of gain K = -1 will give the closed-loop system characteristic equation a root at the origin but that value is of less interest, since we rarely use negative gain values.
- Note that at the loop gain value of 34.65, only system 3 is marginally stable. Systems 1 and 2 show no signs of instability, only improved speed of response and reduced steady-state error.


Response for K = 34.65
2.5
2
1.5
1
0.5
0
-0.5
0
0.5
1
1.5
time (sec)
response: C1 C2 C3
C3
C2
C1

– Root-Locus Interpretation of Stability

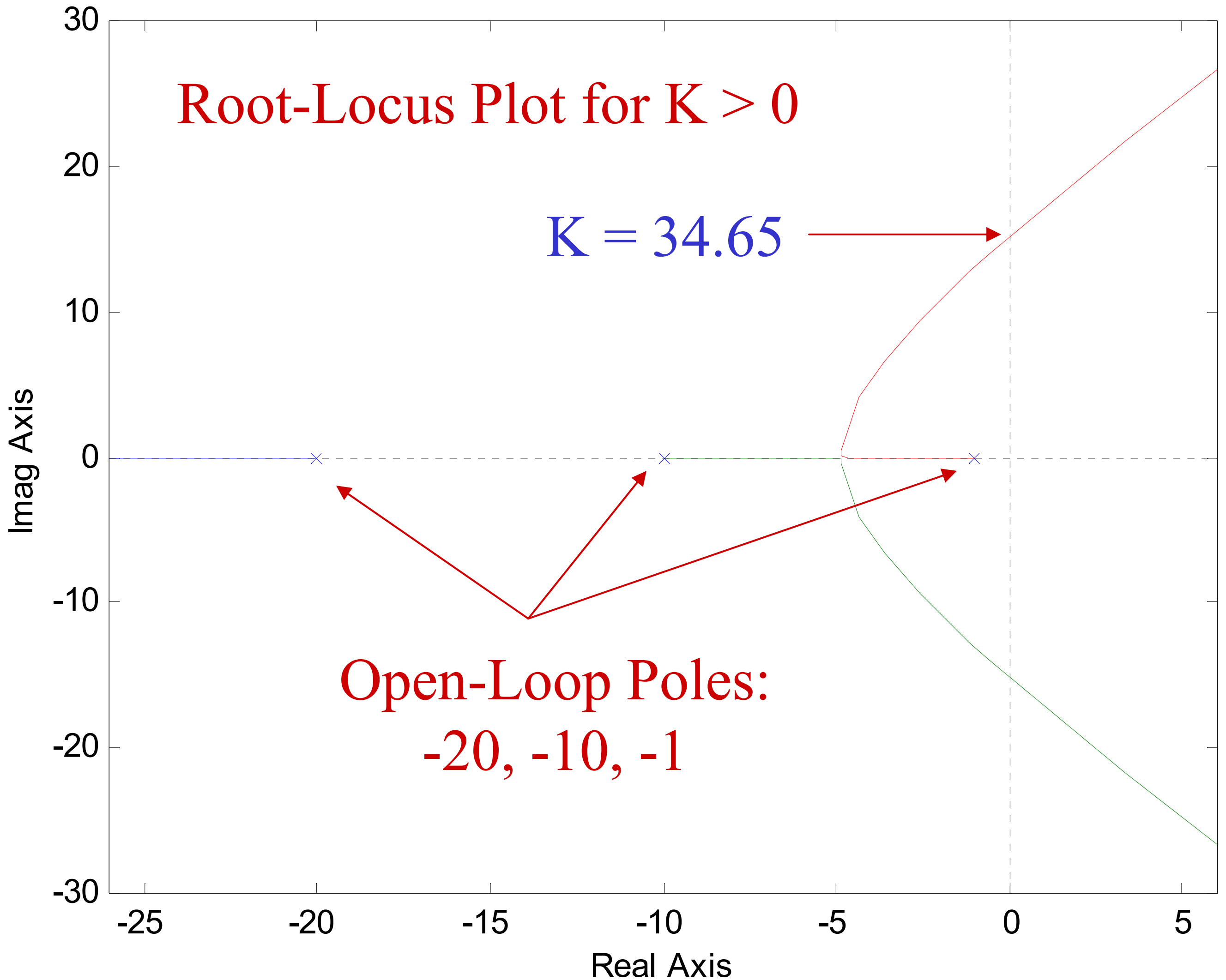
- The open-loop transfer function is:

$$\frac{K}{(\tau_1 s+1)(\tau_2 s+1)(\tau_3 s+1)}$$

$$=\frac{K}{\tau_1\tau_2\tau_3 s^3+(\tau_1\tau_2+\tau_2\tau_3+\tau_1\tau_3)s^2+(\tau_1+\tau_2+\tau_3)s+1}$$

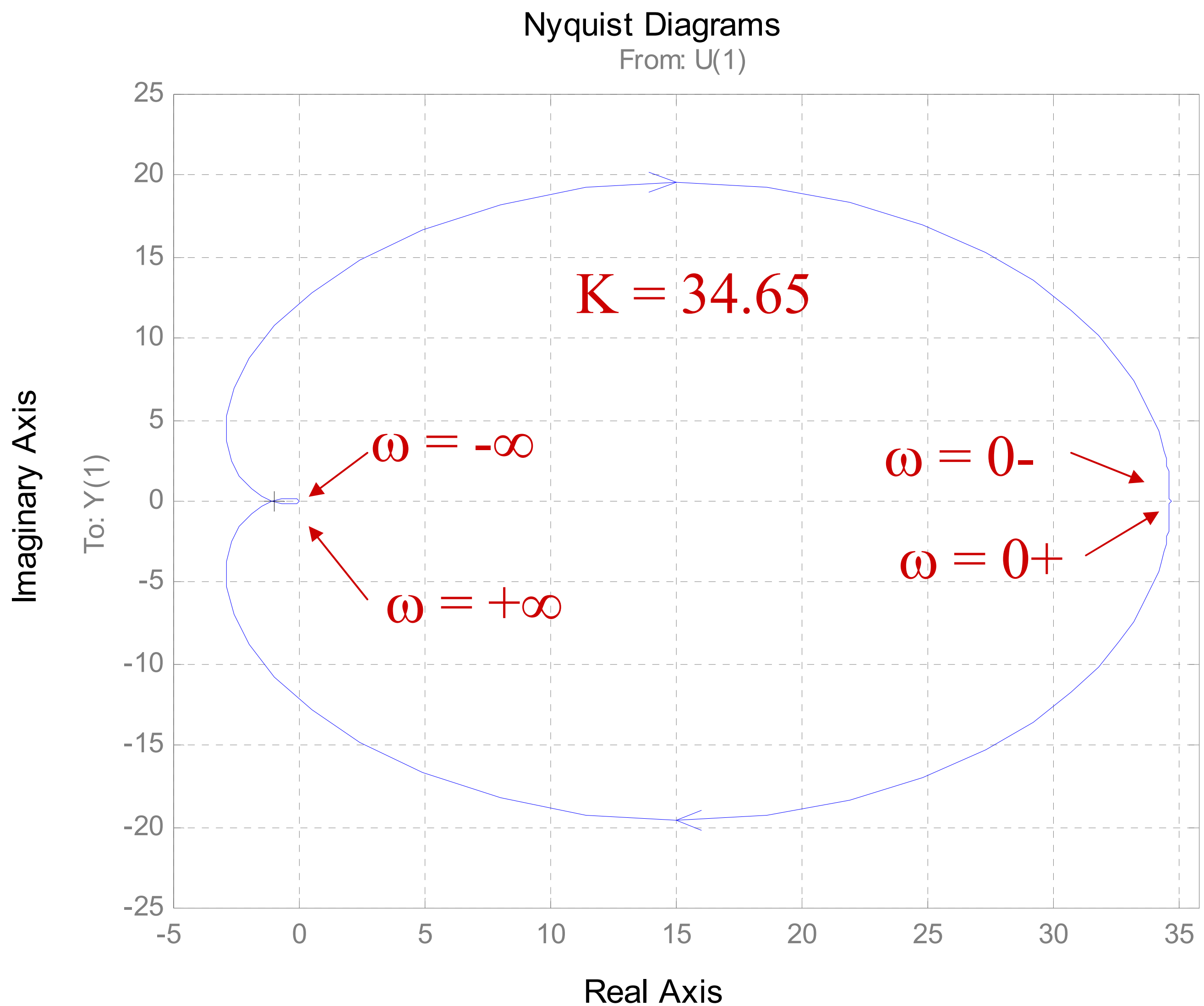
$$=\frac{K}{0.005s^3+0.155s^2+1.15s+1}$$

- The root-locus plot shows that when K = 34.65, the system is marginally stable. For that value of K, the closed-loop poles are at: -31 and ± 15.1658i.

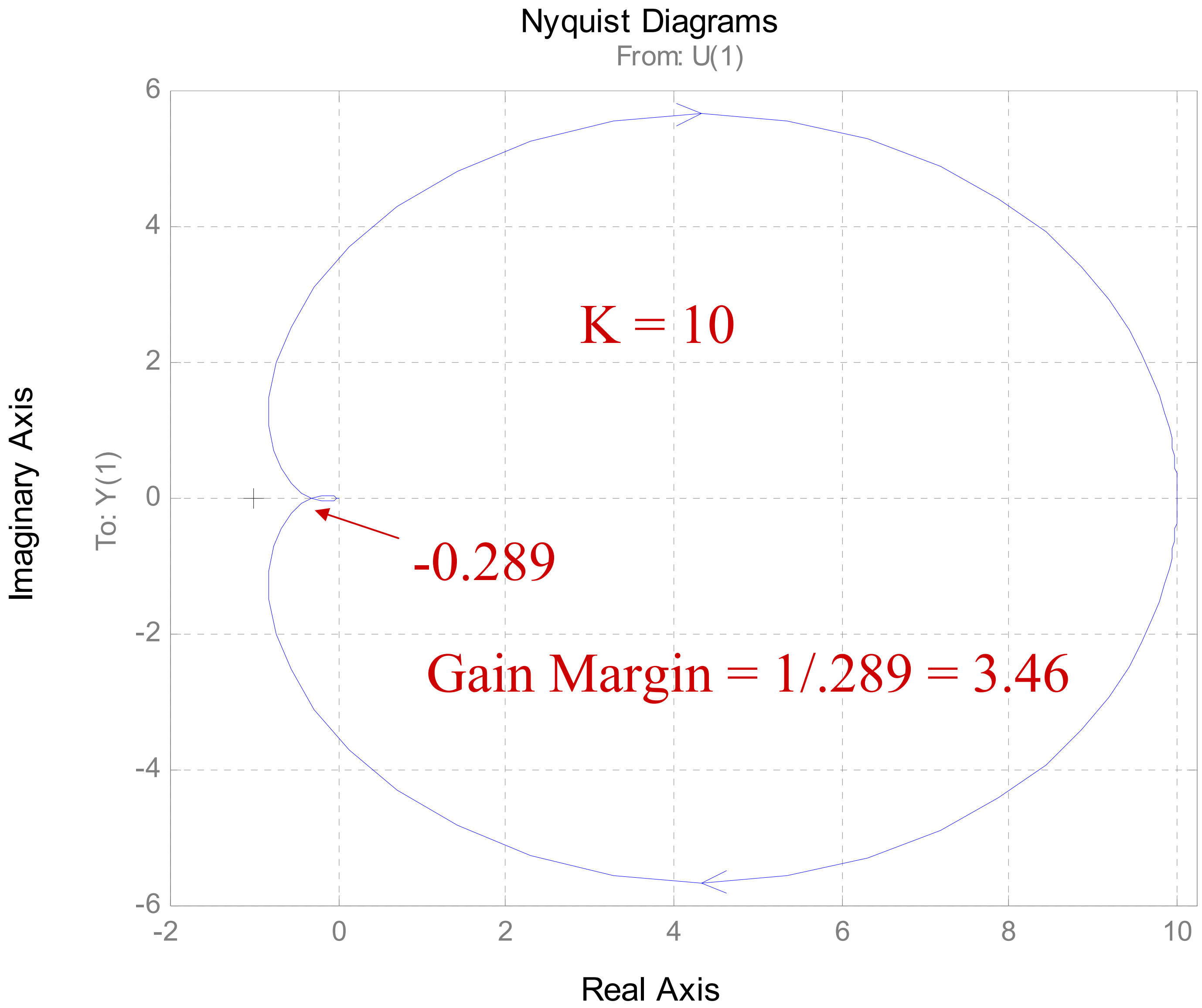

Root-Locus Plot for K > 0
K = 34.65
Open-Loop Poles:
-20, -10, -1
Imag Axis
Real Axis
30
20
10
0
-10
-20
-30
-25
-20
-15
-10
-5
0
5

– Nyquist Stability Criterion

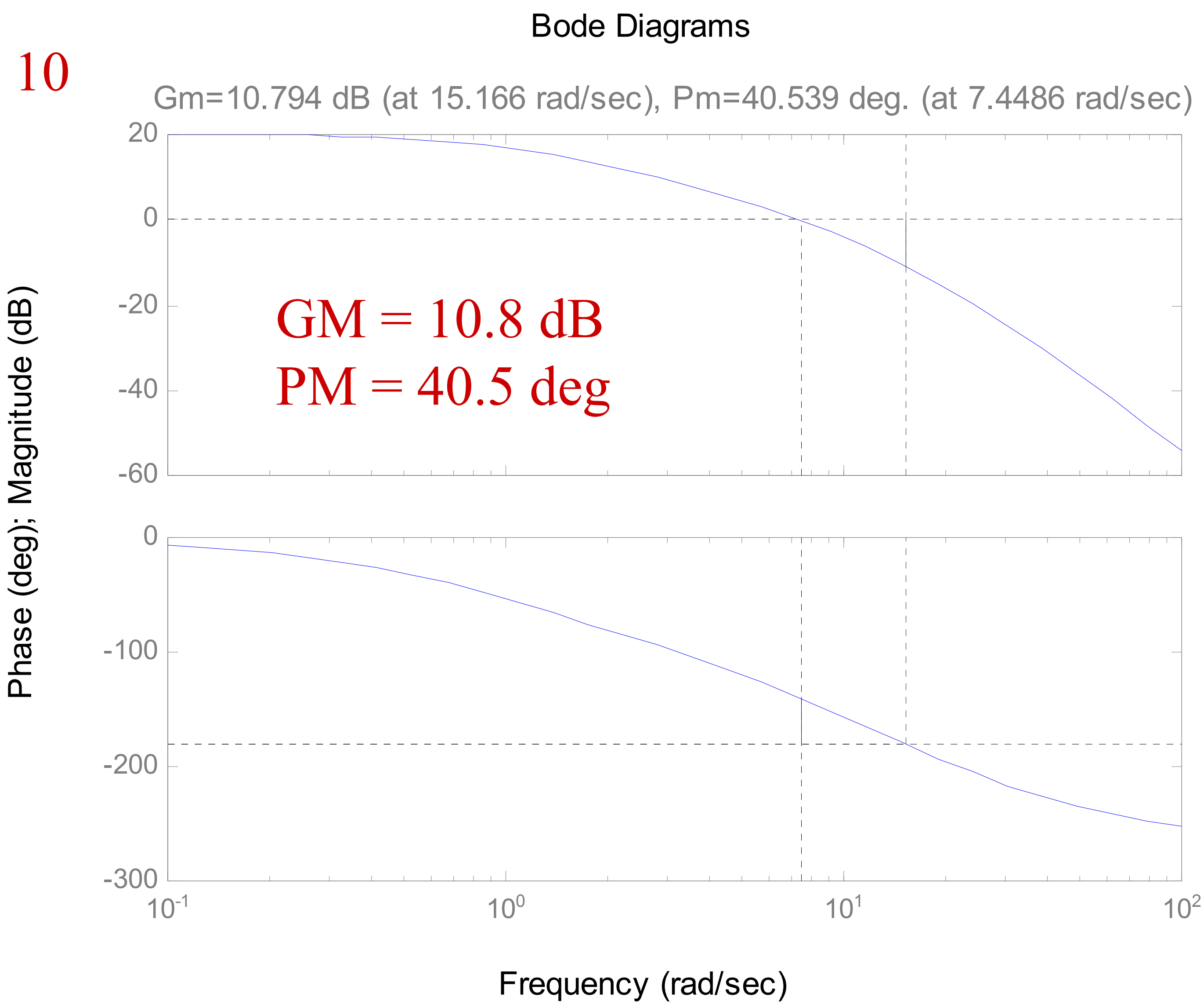
- A polar plot of the open-loop transfer function for the gain K = 34.65 goes through the point –1, indicating marginal stability of the closed-loop system.
- A polar plot of the open-loop transfer function for the gain K = 10 shows a gain margin = 3.46.
- The Bode plots for a gain K = 10 show a gain margin = 10.794 dB = 3.46 and a phase margin = 40.5 degrees.



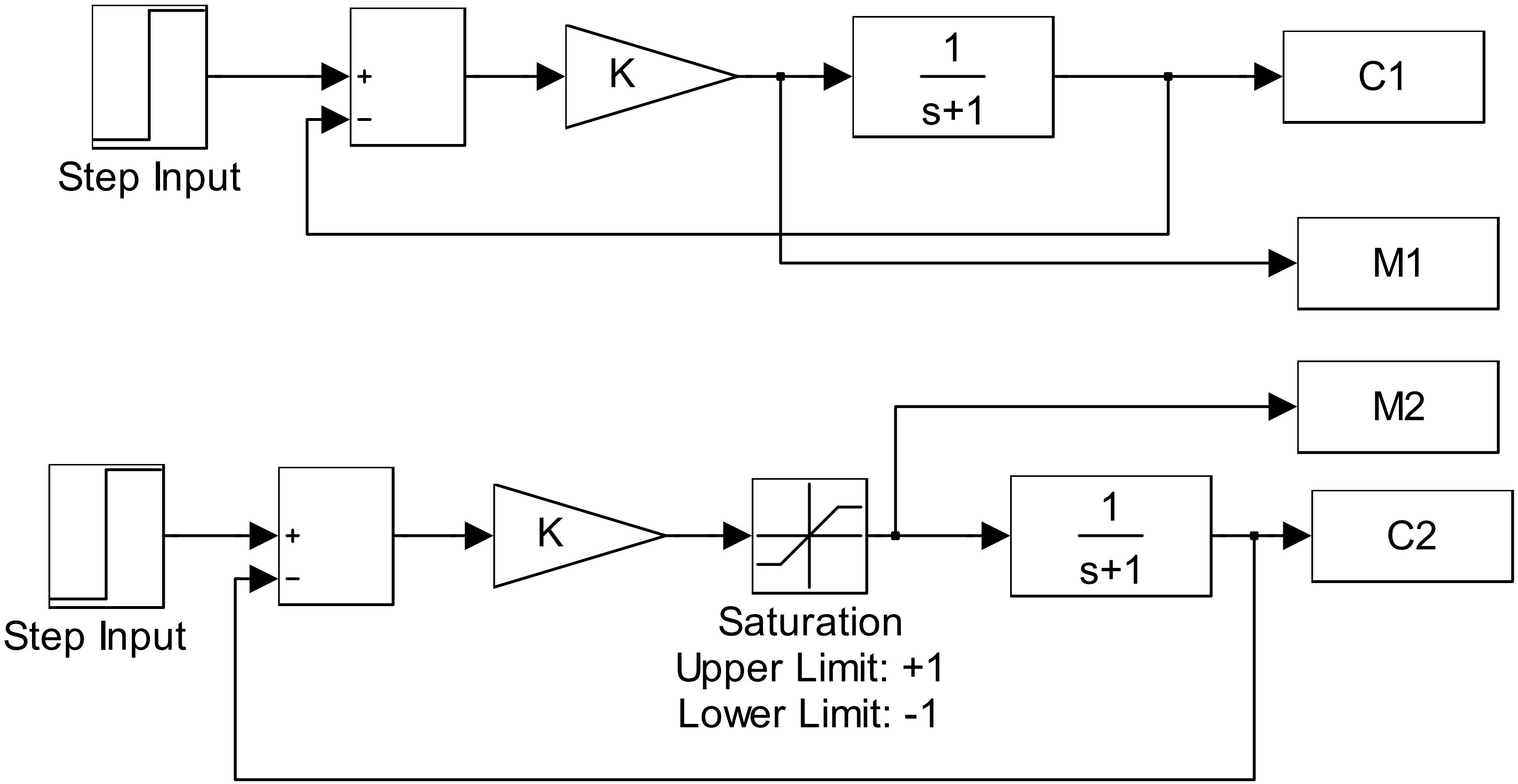
Nyquist Diagrams
From: U(1)
K = 34.65
ω = -∞
ω = +∞
ω = 0-
ω = 0+
Imaginary Axis
To: Y(1)
Real Axis
25
20
15
10
5
0
-5
-10
-15
-20
-25
-5
0
5
10
15
20
25
30
35


Nyquist Diagrams
From: U(1)
Imaginary Axis
To: Y(1)
Real Axis
6
4
2
0
-2
-4
-6
-2
0
2
4
6
8
10
K = 10
-0.289
Gain Margin = 1/.289 = 3.46

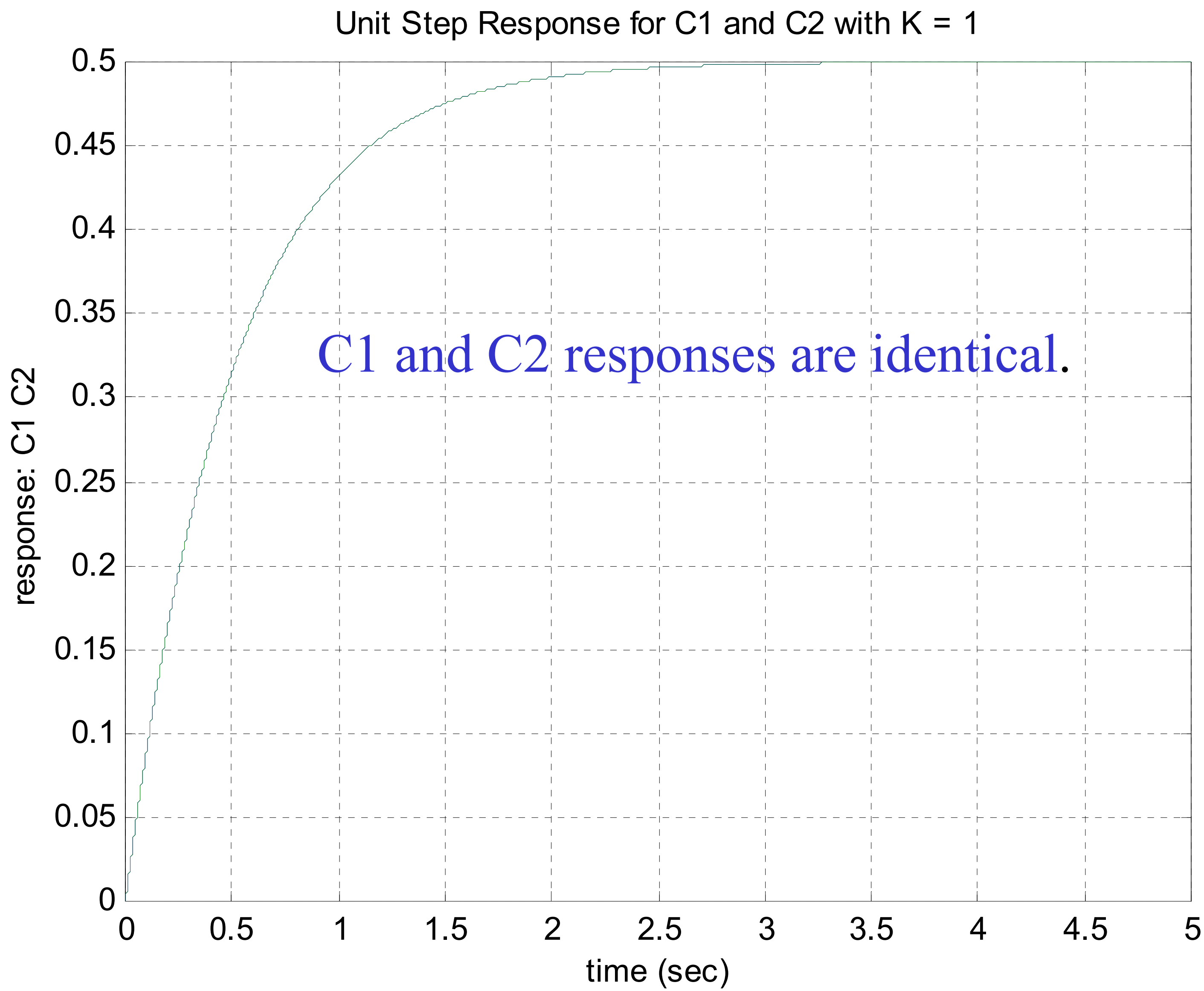
K = 10

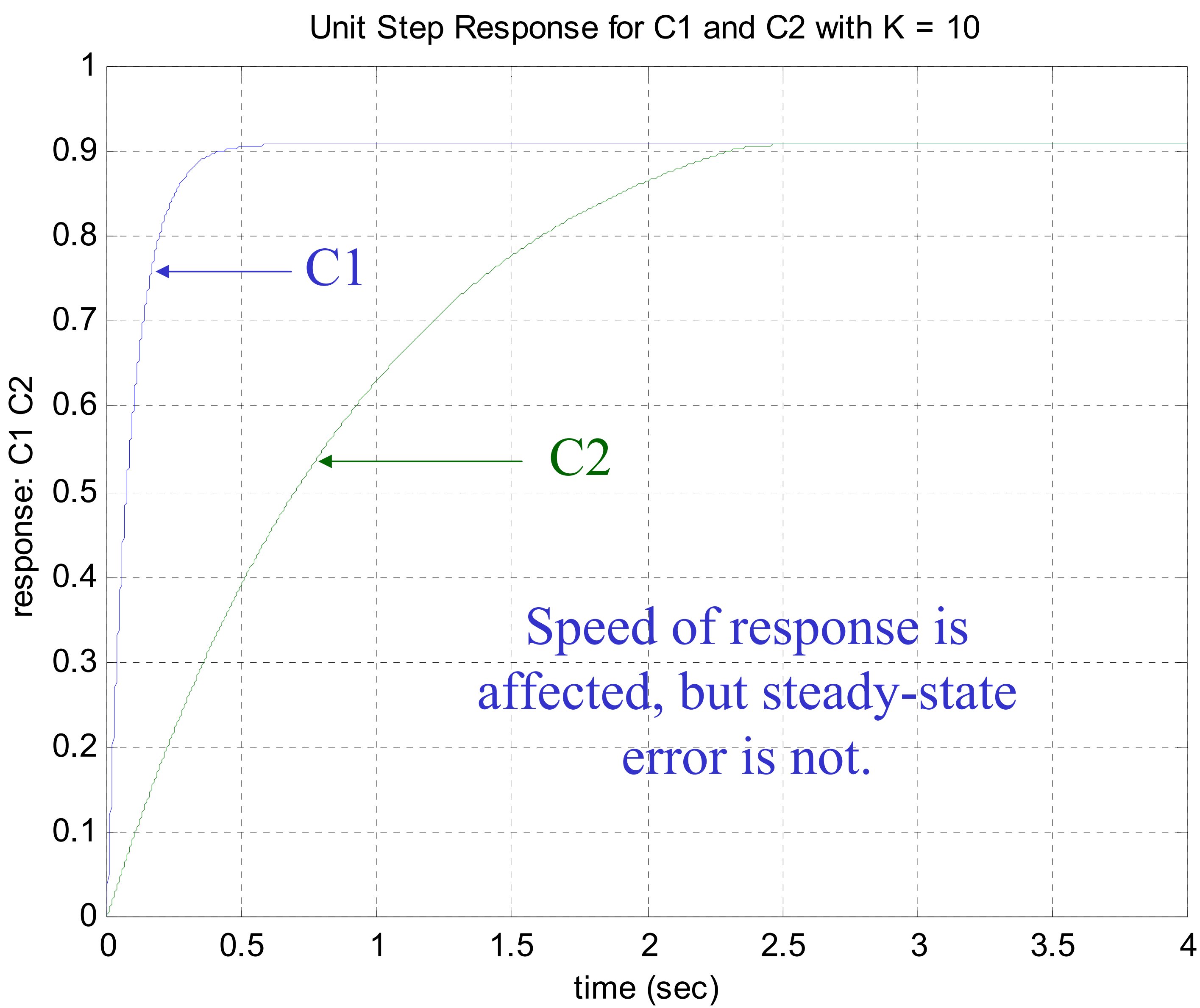

Bode Diagrams
Gm=10.794 dB (at 15.166 rad/sec), Pm=40.539 deg. (at 7.4486 rad/sec)
Phase (deg); Magnitude (dB)
20
0
-20
-40
-60
0
-100
-200
-300
10-1
100
101
102
Frequency (rad/sec)
GM = 10.8 dB
PM = 40.5 deg

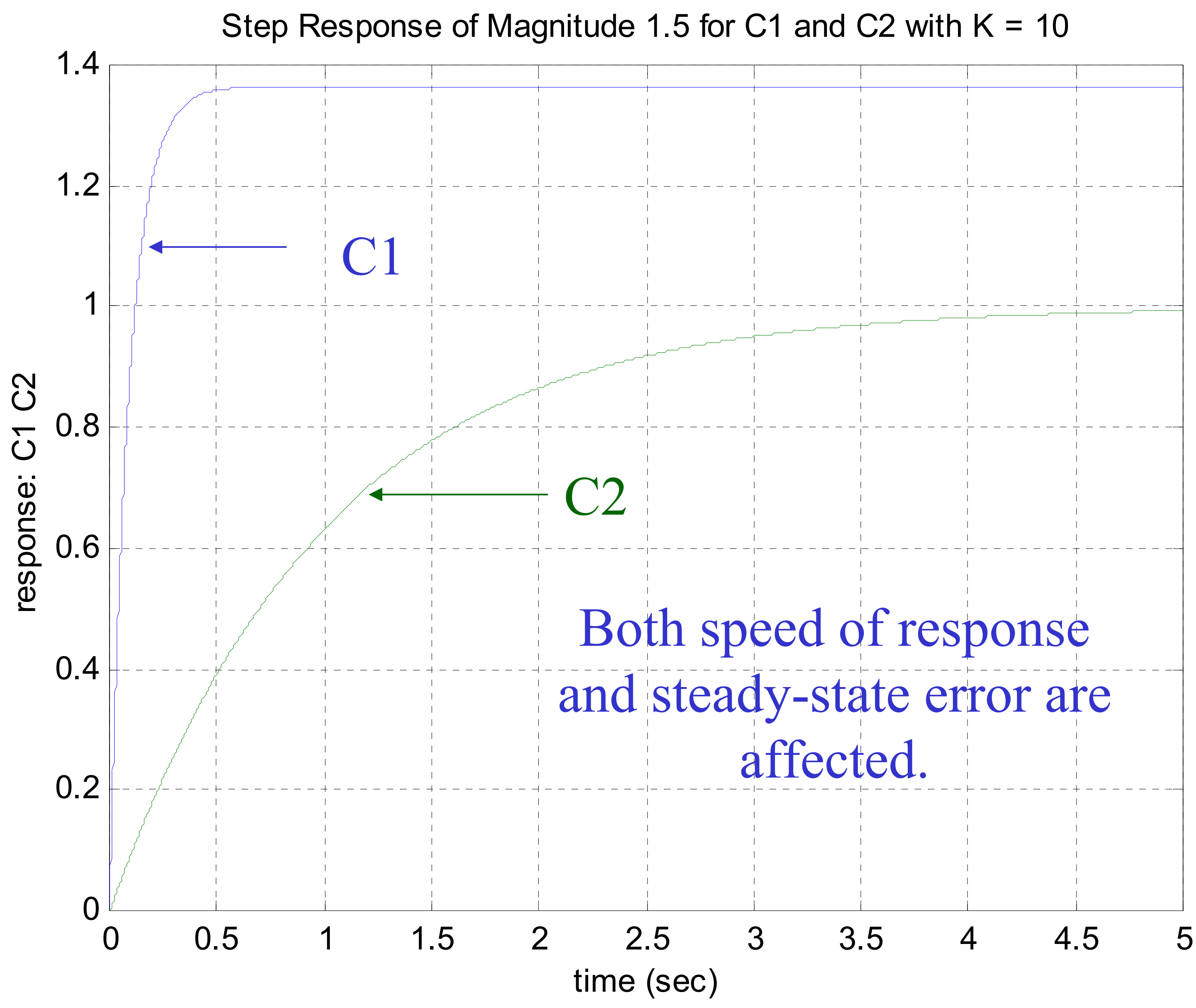
- Saturation
  - The amplifier, electropneumatic transducer, and valve positioner all exhibit saturation, limiting their output when the input becomes too large.
  - All real systems must exhibit such power limitations and one of the consequences is that the closed-loop response speed improvement will not be realized for large signals.
- Exercise:
  - Simulate the following two systems for gains of K = 1, 10 and a unit step input.
  - Examine speed of response and steady-state error.



MatLab / Simulink Diagram


Unit Step Response for C1 and C2 with K = 1
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
response: C1 C2
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
time (sec)
C1 and C2 responses are identical.


Unit Step Response for C1 and C2 with K = 10
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
response: C1 C2
0
0.5
1
1.5
2
2.5
3
3.5
4
time (sec)
C1
C2
Speed of response is affected, but steady-state error is not.


Step Response of Magnitude 1.5 for C1 and C2 with K = 10
response: C1 C2
time (sec)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
0.5
1.5
2
2.5
3
3.5
4
4.5
5
C1
C2
Both speed of response and steady-state error are affected.

- Let's make the system model more realistic by modeling the pneumatic valve positioner as a first-order system:

$$\frac{x_v}{p}(D) = \frac{K_x}{\tau_{vp} D + 1}$$

  – The open-loop system is now second order. The closed-loop system differential equation is now:

$$\left(\frac{D^2}{\omega_n^2} + \frac{2\zeta D}{\omega_n} + 1 \right) h_C = \frac{K}{K+1} h_V + \frac{\tau_{vp} D + 1}{\rho g (K+1)} P_U - \frac{R_f (\tau_{vp} D + 1)}{\rho g (K+1)} Q_U$$

$$\omega_n = \sqrt{\frac{K+1}{\tau_p \tau_{vp}}} \qquad \zeta = \frac{\tau_p + \tau_{vp}}{2\sqrt{\tau_p \tau_{vp} (K+1)}} \qquad K = \frac{1}{\rho g}\left(K_h K_a K_p K_x K_v R_f \right)$$

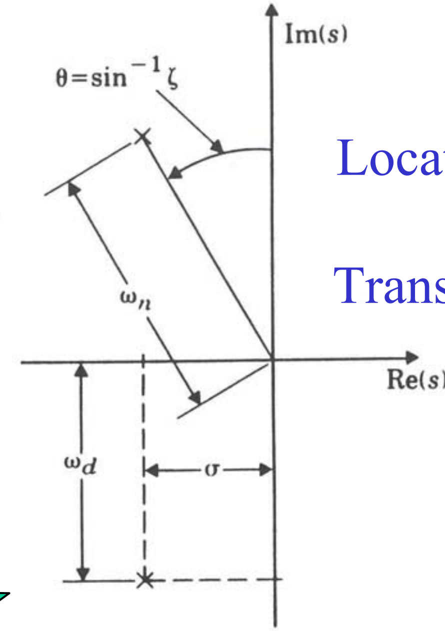
- To get fast response (large ω_n) for given lags τ_p and τ_{vp}, we must increase loop gain K. How does K affect ζ ?
- If $\tau_p = 60$ sec and $\tau_{vp} = 1.0$ sec and we desire $\zeta = 0.6$, what is K? What is ω_n ? Is absolute instability possible with this model? What does the Nyquist plot show as K is increased? What does the root-locus plot show as K is increased?

- Consider Gain Distribution
  - How does gain distribution affect stability and dynamic response of the closed-loop system?
  - Are steady-state errors for disturbances sensitive to gain distribution?
  - Should one optimize the distribution of gain so as to minimize steady-state errors?

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\varsigma\omega_n s + \omega_n^2}$$

$$s_{1,2} = -\varsigma\omega_n \pm i\omega_n\sqrt{1-\varsigma^2}$$

$$s_{1,2} = -\sigma \pm i\omega_d$$



Location of Poles
Of
Transfer Function

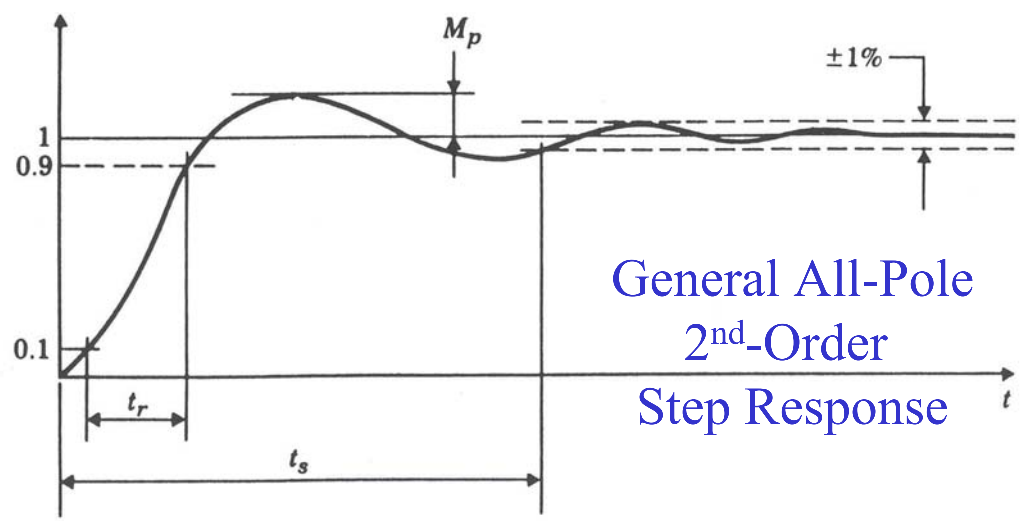
$$y(t) = 1 - e^{-\sigma t}\left(\cos\omega_d t + \frac{\sigma}{\omega_d}\sin\omega_d t\right)$$

$$t_r \approx \frac{1.8}{\omega_n} \quad \text{rise time}$$

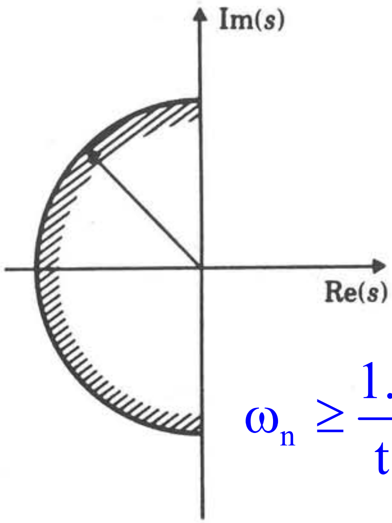
$$t_s \approx \frac{4.6}{\varsigma\omega_n} \quad \text{settling time}$$

$$M_p = e^{\frac{-\pi\varsigma}{\sqrt{1-\varsigma^2}}} \quad (0 \le \zeta < 1) \quad \text{overshoot}$$

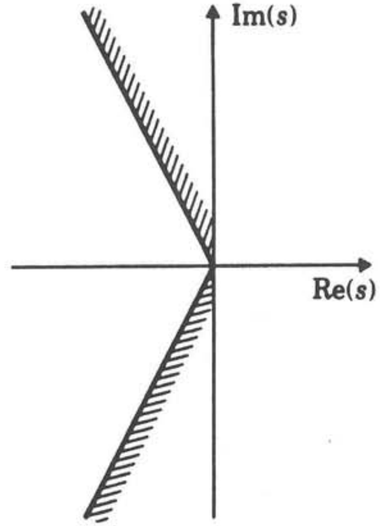
$$\approx \left(1 - \frac{\zeta}{0.6}\right) \qquad (0 \le \zeta \le 0.6)$$

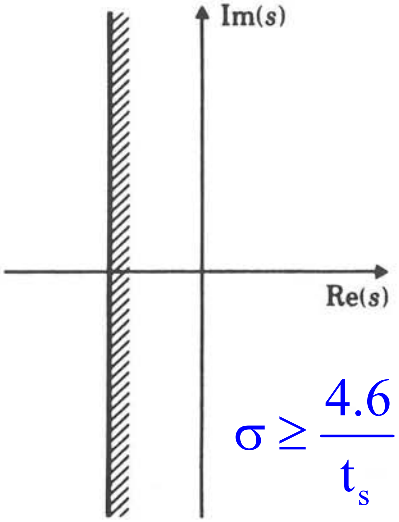


General All-Pole
2nd-Order
Step Response



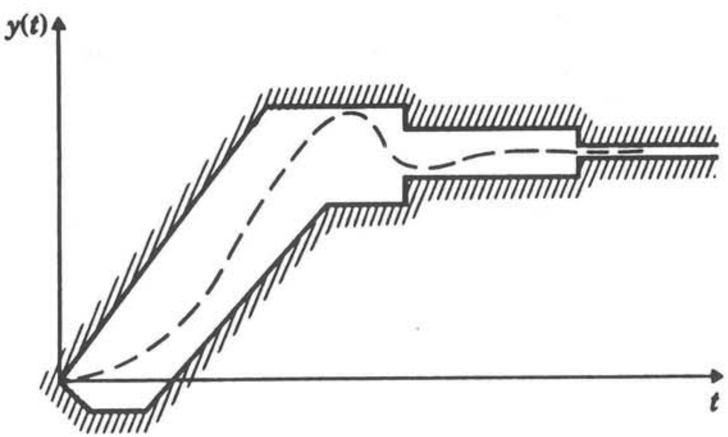
$$\omega_n \geq \frac{1.8}{t_r}$$

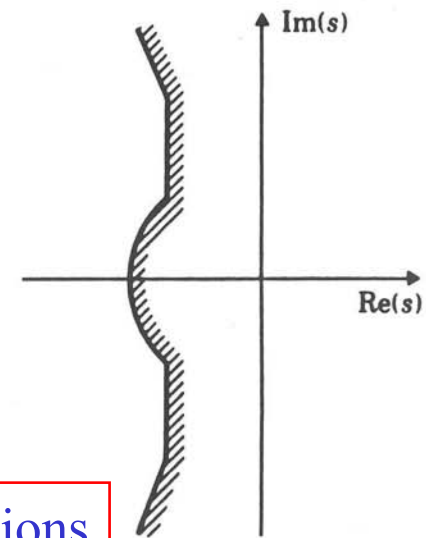




$$\sigma \geq \frac{4.6}{t_s}$$

$$\zeta \geq 0.6\left(1 - M_p\right) \qquad 0 \leq \zeta \leq 0.6$$





Time-Response Specifications vs. Pole-Location Specifications

- <u>Integral Control</u>
  - Consider integral control of this liquid-level process. Replace the amplifier block K_a by K_I/s.
  - The closed-loop system differential equation is:

$$\left[\left(h_V - h_C \right) \frac{K_h K_I K_p K_x K_v}{s} + \frac{1}{R_f} P_U - Q_U \right] \frac{\dfrac{R_f}{\rho g}}{\tau_p s + 1} = h_C$$

$$\left(\tau_p D^2 + D + K \right) h_C = K h_V + \frac{1}{\rho g} D P_U - \frac{R_f}{\rho g} D Q_U$$

$$\left(\tau_p D^2 + D + K \right) h_E = -\left(\tau_p D^2 + D \right) h_V + \frac{1}{\rho g} D P_U - \frac{R_f}{\rho g} D Q_U$$

$$K = \frac{K_h K_I K_p K_x K_v R_f}{\rho g}$$ loop gain

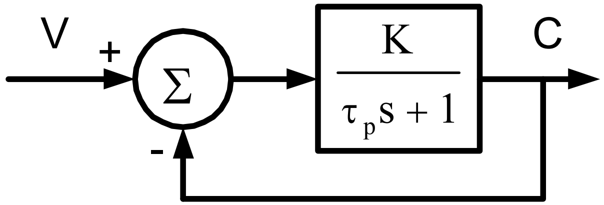
$$h_E = h_V - h_C$$ system error

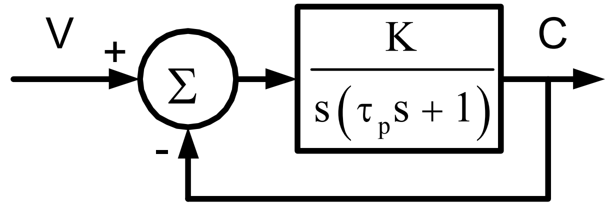
- Note the following:
- Step changes (constant values) of h_V, P_U, and/or Q_U give zero steady-state errors.
- For ramp inputs, we now have constant, nonzero steady-state errors whose magnitudes can be reduced by increasing K.
- The characteristic equation is second order, so define:

$$\omega_n = \sqrt{\frac{K}{\tau_p}} \qquad \zeta = \frac{1}{2\sqrt{K\tau_p}}$$

- If we take τ_p as unavailable for change, we see that an increase in K to gain response speed or decrease ramp steady-state errors will be limited by loss of relative stability (low ζ).

- If we design for a desired ζ, the needed K is easily found and ω_n is then fixed.
- Absolute instability is not predicted; the model is too simple.
- See the comparison between proportional control and integral control: root locus plots and Nyquist plots. Note the destabilizing effects of integral control.

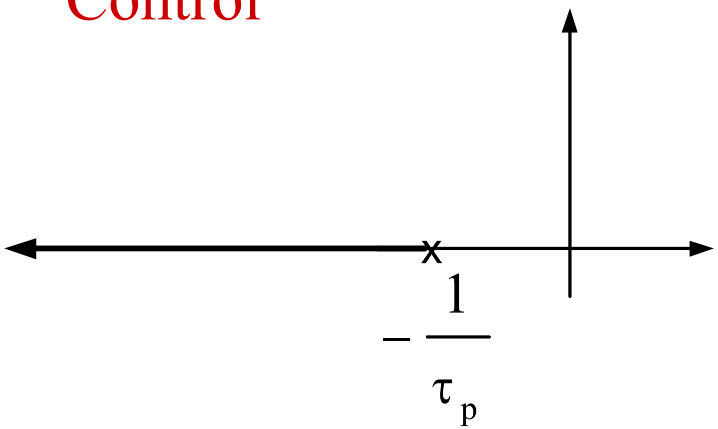


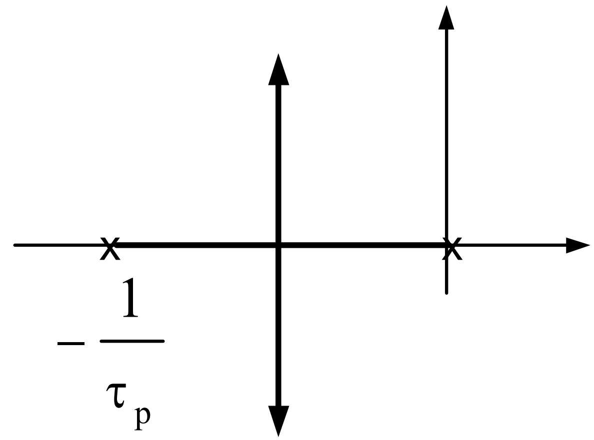


Proportional Control

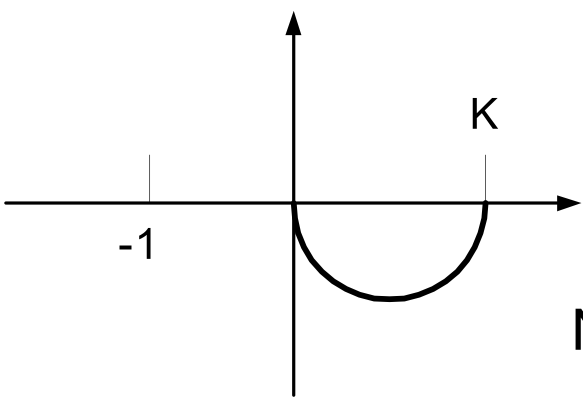
Integral Control

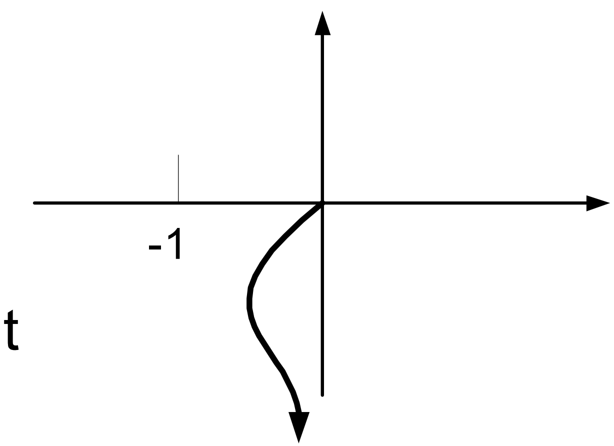
Root Loci





Nyquist Plots





- What if we change the controlled process by closing off the pipe on the left side of the tank. This not only deletes P_U as a disturbance but also causes a significant change in process dynamics.
- Take $R_f = \infty$. The from Conservation of Mass we have:

$$h_C = \frac{1}{A_T s}\left(Q_M - Q_U\right)$$

- The original tank process had self-regulation.
  - If one changes Q_M and/or Q_U the tank will itself in time find a new equilibrium level since the flow through R_f varies with level.

- With R_f not present, the tiniest difference between Q_M and Q_U will cause the tank to completely drain or overflow since it is now an integrator and has lost its self-regulation.
- Even with proportional control, the integrating effect in the process gives zero steady-state error for step commands, but not for disturbances.
- If we substitute integral control to eliminate the Q_U error, the system becomes absolutely unstable.