Principles of Electromechanical Energy Conversion

- Why do we study this?
  - Electromechanical energy conversion theory is the cornerstone for the analysis of electromechanical motion devices.
  - The theory allows us to express the electromagnetic force or torque in terms of the device variables such as the currents and the displacement of the mechanical system.
  - Since numerous types of electromechanical devices are used in motion systems, it is desirable to establish methods of analysis which may be applied to a variety of electromechanical devices rather than just electric machines.

- Plan
  - Establish analytically the relationships which can be used to express the electromagnetic force or torque.
  - Develop a general set of formulas which are applicable to all electromechanical systems with a single mechanical input.
  - Detailed analysis of:
    - Elementary electromagnet
    - Elementary single-phase reluctance machine
    - Windings in relative motion

Lumped Parameters vs. Distributed Parameters

- If the physical size of a device is small compared to the wavelength associated with the signal propagation, the device may be considered lumped and a lumped (network) model employed.

$$\lambda = \frac{v}{f} \quad \left\{ \begin{array}{l} \lambda = \text{wavelength (distance/cycle)} \\ v = \text{velocity of wave propagation (distance/second)} \\ f = \text{signal frequency (Hz)} \end{array} \right.$$

- Consider the electrical portion of an audio system:
  - 20 to 20,000 Hz is the audio range

$$\lambda = \frac{186{,}000 \text{ miles/second}}{20{,}000 \text{ cycles/second}} = 9.3 \text{ miles/cycle}$$

Conservative Force Field

- A force field acting on an object is called *conservative* if the work done in moving the object from one point to another is independent of the path joining the two points.

$$\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$$

$$\int_C \vec{F} \cdot \overrightarrow{dr} \text{ is independent of path if and only if } \nabla \times \vec{F} = 0 \text{ or } \vec{F} = \nabla\phi$$

$$\vec{F} \cdot \overrightarrow{dr} \quad \text{is an exact differential}$$

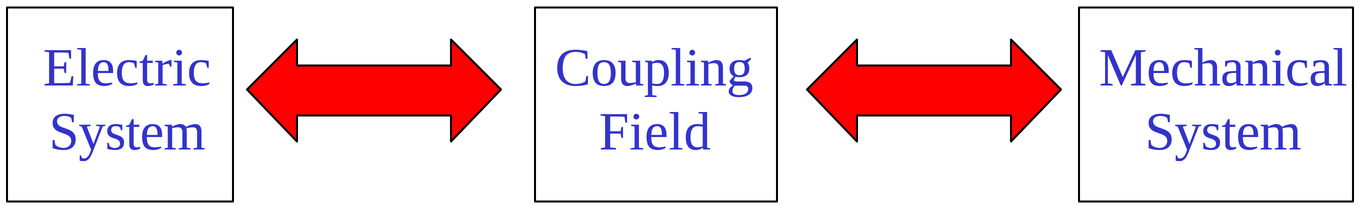
$$F_1 dx + F_2 dy + F_3 dz = d\phi \quad \text{where } \phi(x, y, z)$$

$$\int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} \vec{F} \cdot \overrightarrow{dr} = \int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} d\phi = \phi(x_2, y_2, z_2) - \phi(x_1, y_1, z_1)$$

Energy Balance Relationships

- Electromechanical System
  - Comprises
    - Electric system
    - Mechanical system
    - Means whereby the electric and mechanical systems can interact
  - Interactions can take place through any and all electromagnetic and electrostatic fields which are common to both systems, and energy is transferred as a result of this interaction.
  - Both electrostatic and electromagnetic coupling fields may exist simultaneously and the system may have any number of electric and mechanical subsystems.

- ## Electromechanical System in Simplified Form:



  - Neglect electromagnetic radiation
  - Assume that the electric system operates at a frequency sufficiently low so that the electric system may be considered as a lumped-parameter system

- ## Energy Distribution

$$W_E = W_e + W_{eL} + W_{eS}$$

$$W_M = W_m + W_{mL} + W_{mS}$$

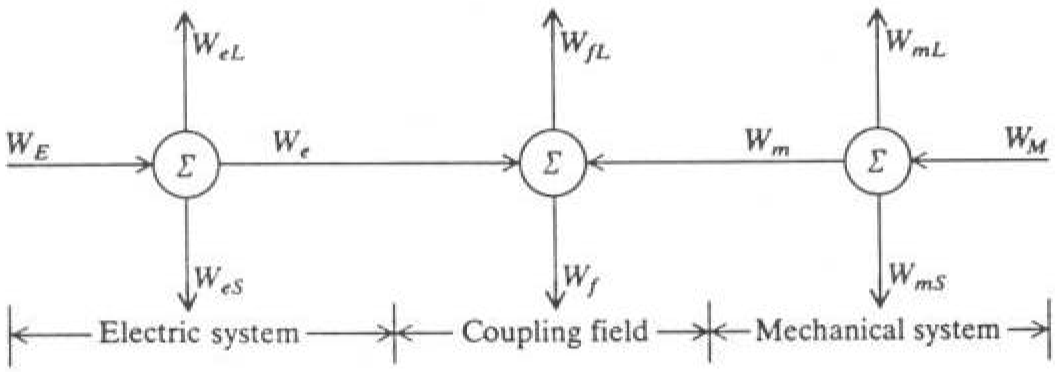
  - W_E = total energy supplied *by* the electric source (+)
  - W_M = total energy supplied *by* the mechanical source (+)

- W_{eS} = energy stored in the electric or magnetic fields which are not coupled with the mechanical system
- W_{eL} = heat loss associated with the electric system, excluding the coupling field losses, which occurs due to:
  - the resistance of the current-carrying conductors
  - the energy dissipated in the form of heat owing to hysteresis, eddy currents, and dielectric losses external to the coupling field
- W_e = energy transferred to the coupling field by the electric system
- W_{mS} = energy stored in the moving member and the compliances of the mechanical system
- W_{mL} = energy loss of the mechanical system in the form of heat due to friction
- W_m = energy transferred to the coupling field by the mechanical system

- $W_F = W_f + W_{fL}$ = total energy transferred to the coupling field
  - W_f = energy stored in the coupling field
  - W_{fL} = energy dissipated in the form of heat due to losses within the coupling field (eddy current, hysteresis, or dielectric losses)
- <u>Conservation of Energy</u>

$$W_f + W_{fL} = \left(W_E - W_{eL} - W_{eS}\right) + \left(W_M - W_{mL} - W_{mS}\right)$$

$$W_f + W_{fL} = W_e + W_m$$



- The actual process of converting electric energy to mechanical energy (or vice versa) is independent of:
  - The loss of energy in either the electric or the mechanical systems (W_{eL} and W_{mL})
  - The energies stored in the electric or magnetic fields which are not in common to both systems (W_{eS})
  - The energies stored in the mechanical system (W_{mS})
- If the losses of the coupling field are neglected, then the field is <u>conservative</u> and $W_f = W_e + W_m$.
- <u>Consider two examples of elementary electromechanical systems</u>
  - Magnetic coupling field
  - Electric field as a means of transferring energy

v = voltage of electric source

f = externally-applied mechanical force

f_e = electromagnetic or electrostatic force

r = resistance of the current-carrying conductor

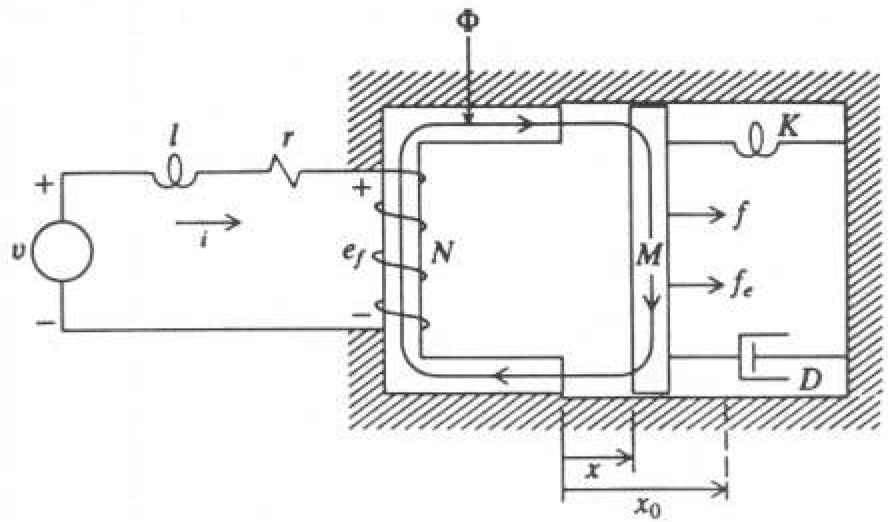
ℓ = inductance of a linear (conservative) electromagnetic system which does not couple the mechanical system

M = mass of moveable member

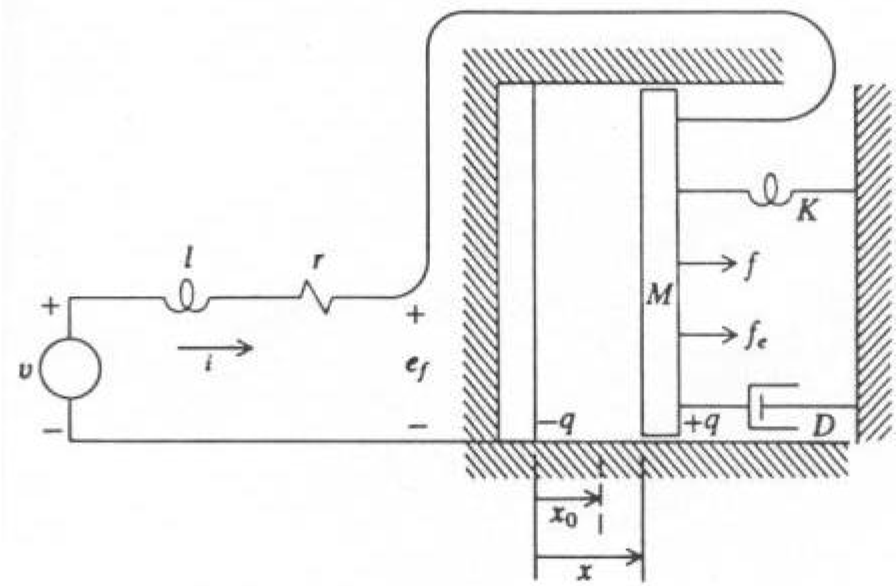
K = spring constant

D = damping coefficient

x_0 = zero force or equilibrium position of the mechanical system ($f_e = 0$, $f = 0$)



Electromechanical System with Magnetic Field



Electromechanical System with Electric Field

$$v = ri + \ell \frac{di}{dt} + e_f$$

voltage equation that describes the electric systems; e_f is the voltage drop due to the coupling field

$$f = M\frac{d^2x}{dt^2} + D\frac{dx}{dt} + K\left(x - x_0\right) - f_e$$

Newton's Law of Motion

$$\left.\begin{aligned} W_E &= \int (vi)\, dt \\ W_M &= \int (f)\, dx = \int \left(f \frac{dx}{dt} \right) dt \end{aligned}\right\}$$

Since power is the time rate of energy transfer, this is the total energy supplied by the electric and mechanical sources

$$\left.\begin{aligned} v &= ri + \ell \frac{di}{dt} + e_f \\ W_E &= \int (vi)\, dt \end{aligned}\right\}$$

$$\begin{aligned} W_E &= r\int \left(i^2\right) dt + \ell \int \left(i \frac{di}{dt} \right) dt + \int \left(e_f i\right) dt \\ &= W_{eL} + W_{eS} + W_e \end{aligned}$$

$$
\left.
\begin{aligned}
f &= M\frac{d^2x}{dt^2} + D\frac{dx}{dt} + K\left(x - x_0\right) - f_e \\
W_M &= \int (f)dx = \int \left(f\frac{dx}{dt} \right) dt
\end{aligned}
\right\}
$$

$$
W_M = M\int \left(\frac{d^2x}{dt^2} \right) dx + D\int \left(\frac{dx}{dt} \right)^2 dt + K\int (x - x_0)dx - \int (f_e)dx
$$

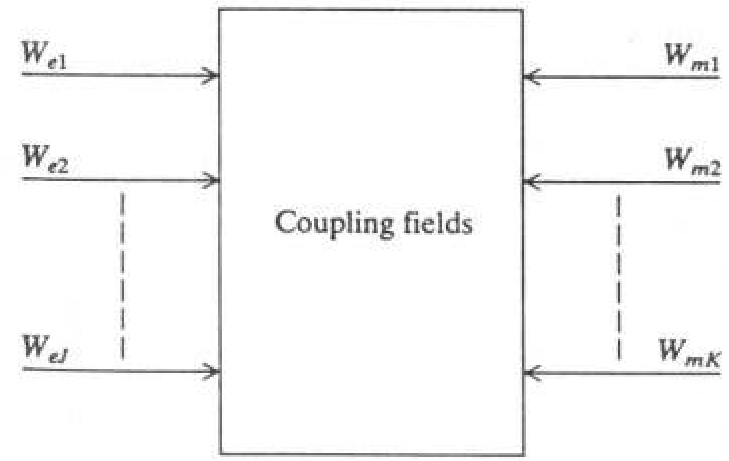
Σ

W_{mS} $\quad$ W_{mL} $\quad$ W_m

$$
W_f = W_e + W_m = \int (e_f i)dt - \int (f_e)dx
$$

total energy transferred to the coupling field

- These equations may be readily extended to include an electromechanical system with any number of electrical and mechanical inputs and any number of coupling fields.



- We will consider devices with only one mechanical input, but with possibly multiple electric inputs. In all cases, however, the multiple electric inputs have a common coupling field.

$$W_f = \sum_{j=1}^{J} W_{ej} + \sum_{k=1}^{K} W_{mk}$$

Total energy supplied to the coupling field

$$\sum_{j=1}^{J} W_{ej} = \int \sum_{j=1}^{J} e_{fj} i_j dt$$

Total energy supplied to the coupling field from the electric inputs

$$\sum_{k=1}^{K} W_{mk} = -\int \sum_{k=1}^{K} f_{ek} dx_k$$

Total energy supplied to the coupling field from the mechanical inputs

$$W_f = \int \sum_{j=1}^{J} e_{fj} i_j dt - \int f_e dx$$

$$dW_f = \sum_{j=1}^{J} e_{fj} i_j dt - f_e dx$$

With one mechanical input and multiple electric inputs, the energy supplied to the coupling field, in both integral and differential form

Energy in Coupling Field

- We need to derive an expression for the energy stored in the coupling field before we can solve for the electromagnetic force f_e.
- We will neglect all losses associated with the electric or magnetic coupling field, whereupon the field is assumed to be conservative and the energy stored therein is a function of the state of the electrical and mechanical variables and not the manner in which the variables reached that state.
- This assumption is not as restrictive as it might first appear.

- The ferromagnetic material is selected and arranged in laminations so as to minimize the hysteresis and eddy current losses.
- Nearly all of the energy stored in the coupling field is stored in the air gap of the electromechanical device. Air is a conservative medium and all of the energy stored therein can be returned to the electric or mechanical systems.

- We will take advantage of the conservative field assumption in developing a mathematical expression for the field energy. We will fix mathematically the position of the mechanical system associated with the coupling field and then excite the electric system with the displacement of the mechanical system held fixed.

- During the excitation of the electric inputs, $dx = 0$, hence, W_m is zero even though electromagnetic and electrostatic forces occur.
- Therefore, with the displacement held fixed, the energy stored in the coupling field during the excitation of the electric inputs is equal to the energy supplied to the coupling field by the electric inputs.
- With $dx = 0$, the energy supplied from the electric system is:

$$W_f = \int \sum_{j=1}^{J} e_{fj} i_j dt - \int f_e \, \overset{0}{dx}$$

$$W_f = \int \sum_{j=1}^{J} e_{fj} i_j dt$$

- For a singly excited electromagnetic system:

$$e_f = \frac{d\lambda}{dt}$$

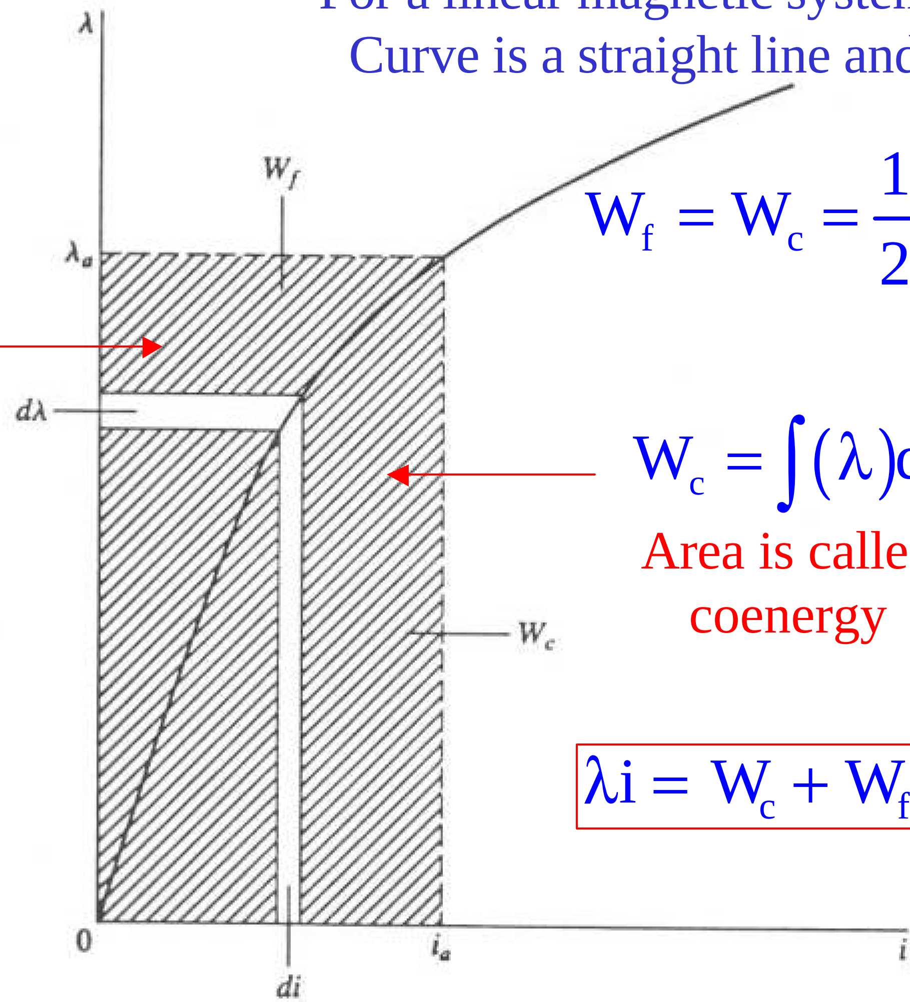
$$W_f = \int (i)\,d\lambda \quad \text{with } dx = 0$$

For a linear magnetic system: Curve is a straight line and

$$W_f = W_c = \frac{1}{2}\lambda i$$

$$W_f = \int (i)\,d\lambda$$

Area represents energy stored in the field at the instant when $\lambda = \lambda_a$ and $i = i_a$.



$$W_c = \int (\lambda)\,di$$

Area is called coenergy

$$\boxed{\lambda i = W_c + W_f}$$

Graph

Stored energy and coenergy in a magnetic field of a singly excited electromagnetic device

- The $l i$ relationship need not be linear, it need only be single-valued, a property which is characteristic to a conservative or lossless field.
- Also, since the coupling field is conservative, the energy stored in the field with $l = l_a$ and $i = i_a$ is independent of the excursion of the electrical and mechanical variables before reaching this state.
- The displacement x defines completely the influence of the mechanical system upon the coupling field; however, since l and i are related, only one is needed in addition to x in order to describe the state of the electromechanical system.

- If *i* and *x* are selected as the independent variables, it is convenient to express the field energy and the flux linkages as

$$W_f = W_f(i, x)$$

$$\lambda = \lambda(i, x)$$

$$d\lambda = \frac{\partial \lambda(i, x)}{\partial i} di + \frac{\partial \lambda(i, x)}{\partial x} dx$$

$$d\lambda = \frac{\partial \lambda(i, x)}{\partial i} di \quad \text{with } dx = 0$$

$$W_f = \int (i) d\lambda = \int i \frac{\partial \lambda(i, x)}{\partial i} di = \int_0^i \xi \frac{\partial \lambda(\xi, x)}{\partial \xi} d\xi$$

Energy stored in the field of a singly excited system

- The coenergy in terms of *i* and *x* may be evaluated as

$$W_c(i,x)=\int \lambda(i,x)\,di=\int_0^i \lambda(\xi,x)\,d\xi$$

- For a linear electromagnetic system, the *l i* plots are straightline relationships. Thus, for the singly excited magnetically linear system, $\lambda(i,x)=L(x)i$, where *L(x)* is the inductance.
- Let's evaluate $W_f(i,x)$.

$$d\lambda=\frac{\partial\lambda(i,x)}{\partial i}di \quad \text{with } dx=0$$

$$d\lambda=L(x)\,di$$

$$W_f(i,x)=\int_0^i \xi L(x)\,d\xi=\frac{1}{2}L(x)i^2$$

- The field energy is a state function and the expression describing the field energy in terms of the state variables is valid regardless of the variations in the system variables.
- W_f expresses the field energy regardless of the variations in *L(x)* and *i*. The fixing of the mechanical system so as to obtain an expression for the field energy is a mathematical convenience and not a restriction upon the result.

$$W_f(i,x) = \int_0^i \xi L(x)\,d\xi = \frac{1}{2}L(x)i^2$$

- In the case of a multiexcited electromagnetic system, an expression for the field energy may be obtained by evaluating the following relation with dx = 0:

$$W_f = \int \sum_{j=1}^{J} i_j d\lambda_j$$

- Since the coupling field is considered conservative, this expression may be evaluated independent of the order in which the flux linkages or currents are brought to their final values.
- Let's consider a doubly excited electric system with one mechanical input.

$$W_f(i_1, i_2, x) = \int \left[i_1 d\lambda_1(i_1, i_2, x) + i_2 d\lambda_2(i_1, i_2, x) \right] \quad \text{with } dx = 0$$

- The result is:

$$W_f\left(i_1, i_2, x\right) = \int_0^{i_1} \xi \frac{\partial \lambda_1\left(\xi, 0, x\right)}{\partial \xi} d\xi +$$

$$\int_0^{i_2} \left[i_1 \frac{\partial \lambda_1\left(i_1, \xi, x\right)}{\partial \xi} + \xi \frac{\partial \lambda_2\left(i_1, \xi, x\right)}{\partial \xi} \right] d\xi$$

- The first integral results from the first step of the evaluation with i_1 as the variable of integration and with $i_2 = 0$ and $di_2 = 0$. The second integral comes from the second step of the evaluation with i_1 equal to its final value ($di_1 = 0$) and i_2 as the variable of integration. The order of allowing the currents to reach their final state is irrelevant.

- Let’s now evaluate the energy stored in a magnetically linear electromechanical system with two electrical inputs and one mechanical input.

$$\lambda_1(i_1, i_2, x) = L_{11}(x)i_1 + L_{12}(x)i_2$$
$$\lambda_2(i_1, i_2, x) = L_{21}(x)i_1 + L_{22}(x)i_2$$

- The self-inductances $L_{11}(x)$ and $L_{22}(x)$ include the leakage inductances.
- With the mechanical displacement held constant (dx = 0):

$$d\lambda_1(i_1, i_2, x) = L_{11}(x)di_1 + L_{12}(x)di_2$$
$$d\lambda_2(i_1, i_2, x) = L_{21}(x)di_1 + L_{22}(x)di_2$$

- Substitution into:

$$W_f\left(i_1,i_2,x\right)=\int_0^{i_1}\xi\frac{\partial\lambda_1\left(\xi,0,x\right)}{\partial\xi}d\xi+$$

$$\int_0^{i_2}\left[i_1\frac{\partial\lambda_1\left(i_1,\xi,x\right)}{\partial\xi}+\xi\frac{\partial\lambda_2\left(i_1,\xi,x\right)}{\partial\xi}\right]d\xi$$

- Yields:

$$W_f\left(i_1,i_2,x\right)=\int_0^{i_1}\xi L_{11}\left(x\right)d\xi+\int_0^{i_2}\left[i_1L_{12}\left(x\right)+\xi L_{22}\left(x\right)\right]d\xi$$

$$=\frac{1}{2}L_{11}\left(x\right)i_1^2+L_{12}\left(x\right)i_1i_2+\frac{1}{2}L_{22}\left(x\right)i_2^2$$

- It follows that the total field energy of a linear electromagnetic system with J electric inputs may be expressed as:

$$W_f\left(i_1,\ldots,i_j,x\right)=\frac{1}{2}\sum_{p=1}^{J}\sum_{q=1}^{J}L_{pq}i_pi_q$$

Electromagnetic and Electrostatic Forces

- Energy Balance Equation:

$$W_f = \int \sum_{j=1}^{J} e_{fj} i_j dt - \int f_e dx$$

$$dW_f = \sum_{j=1}^{J} e_{fj} i_j dt - f_e dx$$

$$f_e dx = \sum_{j=1}^{J} e_{fj} i_j dt - dW_f$$

- To obtain an expression for f_e, it is first necessary to express W_f and then take its total derivative. The total differential of the field energy is required here.

- The force or torque in any electromechanical system may be evaluated by employing: $dW_f = dW_e + dW_m$
- We will derive the force equations for electro-mechanical systems with one mechanical input and *J* electrical inputs.
- For an electromagnetic system: $f_e dx = \sum_{j=1}^{J} i_j d\lambda_j - dW_f$
- Select i_j and x as independent variables: $W_f = W_f\left(\vec{i}, x\right)$ $\lambda_j = \lambda_j\left(\vec{i}, x\right)$

$$dW_f = \sum_{j=1}^{J}\left[\frac{\partial W_f\left(\vec{i}, x\right)}{\partial i_j} di_j\right] + \frac{\partial W_f\left(\vec{i}, x\right)}{\partial x} dx$$

$$d\lambda_j = \sum_{n=1}^{J}\left[\frac{\partial \lambda_j\left(\vec{i}, x\right)}{\partial i_n} di_n\right] + \frac{\partial \lambda_j\left(\vec{i}, x\right)}{\partial x} dx$$

- The summation index n is used so as to avoid confusion with the subscript j since each $d l_j$ must be evaluated for changes in all currents to account for mutual coupling between electric systems.
- Substitution:

$$dW_f = \sum_{j=1}^{J}\left[\frac{\partial W_f\left(\vec{i},x\right)}{\partial i_j}di_j\right] + \frac{\partial W_f\left(\vec{i},x\right)}{\partial x}dx$$

$$d\lambda_j = \sum_{n=1}^{J}\left[\frac{\partial \lambda_j\left(\vec{i},x\right)}{\partial i_n}di_n\right] + \frac{\partial \lambda_j\left(\vec{i},x\right)}{\partial x}dx$$

into

$$f_e dx = \sum_{j=1}^{J} i_j d\lambda_j - dW_f$$

- Result:

$$f_e\left(\vec{i},x\right)dx = \sum_{j=1}^{J} i_j \left\{ \sum_{n=1}^{J} \left[\frac{\partial \lambda j\left(\vec{i},x\right)}{\partial i_n} di_n \right] + \frac{\partial \lambda_j\left(\vec{i},x\right)}{\partial x} dx \right\}$$

$$-\sum_{j=1}^{J} \left[\frac{\partial W_f\left(\vec{i},x\right)}{\partial i_j} di_j \right] + \frac{\partial W_f\left(\vec{i},x\right)}{\partial x} dx$$

$$f_e\left(\vec{i},x\right)dx = \left\{ \sum_{j=1}^{J} \left[i_j \frac{\partial \lambda_j\left(\vec{i},x\right)}{\partial x} \right] - \frac{\partial W_f\left(\vec{i},x\right)}{\partial x} \right\} dx$$

$$+\sum_{j=1}^{J} \left\{ i_j \sum_{n=1}^{J} \left[\frac{\partial \lambda_j\left(\vec{i},x\right)}{\partial i_n} di_n \right] - \frac{\partial W_f\left(\vec{i},x\right)}{\partial i_j} di_j \right\}$$

- This equation is satisfied provided that:

$$f_e\left(\vec{i},x\right)=\sum_{j=1}^{J}\left[i_j\frac{\partial\lambda_j\left(\vec{i},x\right)}{\partial x}\right]-\frac{\partial W_f\left(\vec{i},x\right)}{\partial x}$$

$$0=\sum_{j=1}^{J}\left\{i_j\sum_{n=1}^{J}\left[\frac{\partial\lambda_j\left(\vec{i},x\right)}{\partial i_n}di_n\right]-\frac{\partial W_f\left(\vec{i},x\right)}{\partial i_j}di_j\right\}$$

- The first equation can be used to evaluate the force on the mechanical system with *i* and *x* selected as independent variables.

- We can incorporate an expression for coenergy and obtain a second force equation:

$$W_c = \sum_{j=1}^{J} i_j \lambda_j - W_f$$

- Since *i* and *x* are independent variables, the partial derivative with respect to *x* is:

$$\frac{\partial W_c(\vec{i}, x)}{\partial x} = \sum_{j=1}^{J}\left[i_j \frac{\partial \lambda_j(\vec{i}, x)}{\partial x} \right] - \frac{\partial W_f(\vec{i}, x)}{\partial x}$$

- Substitution:

$$f_e(\vec{i}, x) = \sum_{j=1}^{J}\left[i_j \frac{\partial \lambda_j(\vec{i}, x)}{\partial x} \right] - \frac{\partial W_f(\vec{i}, x)}{\partial x} = \frac{\partial W_c(\vec{i}, x)}{\partial x}$$

- Note:
  - Positive f_e and positive dx are in the same direction
  - If the magnetic system is linear, $W_c = W_f$.
- Summary:

$$f_e\left(\vec{i},x\right) = \sum_{j=1}^{J}\left[i_j \frac{\partial \lambda_j\left(\vec{i},x\right)}{\partial x}\right] - \frac{\partial W_f\left(\vec{i},x\right)}{\partial x}$$

$$f_e\left(\vec{i},x\right) = \frac{\partial W_c\left(\vec{i},x\right)}{\partial x}$$

$$T_e\left(\vec{i},\theta\right) = \sum_{j=1}^{J}\left[i_j \frac{\partial \lambda_j\left(\vec{i},\theta\right)}{\partial \theta}\right] - \frac{\partial W_f\left(\vec{i},\theta\right)}{\partial \theta}$$

$$f_e \Rightarrow T_e$$

$$x \Rightarrow \theta$$

$$T_e\left(\vec{i},\theta\right) = \frac{\partial W_c\left(\vec{i},\theta\right)}{\partial \theta}$$

- By a similar procedure, force equations may be derived with flux linkages λ_1, …, λ_j of the J windings and x as independent variables. The relations, given without proof, are:

$$f_e\left(\vec{\lambda}, x\right) = -\sum_{j=1}^{J}\left[\lambda_j \frac{\partial i_j\left(\vec{\lambda}, x\right)}{\partial x}\right] + \frac{\partial W_c\left(\vec{\lambda}, x\right)}{\partial x}$$

$$f_e\left(\vec{\lambda}, x\right) = -\frac{\partial W_f\left(\vec{\lambda}, x\right)}{\partial x}$$

$$T_e\left(\vec{\lambda}, \theta\right) = -\sum_{j=1}^{J}\left[\lambda_j \frac{\partial i_j\left(\vec{\lambda}, \theta\right)}{\partial \theta}\right] + \frac{\partial W_c\left(\vec{\lambda}, \theta\right)}{\partial \theta}$$

$$T_e\left(\vec{\lambda}, \theta\right) = -\frac{\partial W_f\left(\vec{\lambda}, \theta\right)}{\partial \theta}$$

- One may prefer to determine the electromagnetic force or torque by starting with the relationship

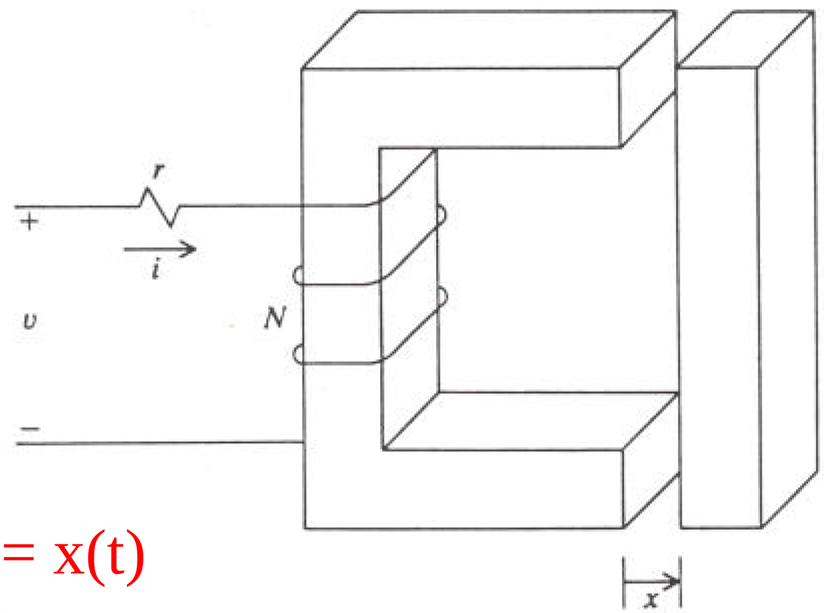
$$dW_f = dW_e + dW_m$$

rather than by selecting a formula.

- Example:
  - Given: $\lambda = \left[1 + a(x)\right] i^2$
  - Find $f_e(i,x)$

Elementary Electromagnet

- The system consists of:
  - stationary core with a winding of N turns
  - block of magnetic material is free to slide relative to the stationary member



x = x(t)

$$v = ri + \frac{d\lambda}{dt}$$ voltage equation that describes the electric system

$$\left.\begin{aligned} &\lambda = N\phi \\ &\phi = \phi_\ell + \phi_m \\ &\phi_\ell = \text{ leakage flux} \\ &\phi_m = \text{ magnetizing flux} \end{aligned}\right\}$$

flux linkages

(the magnetizing flux is common to both stationary and rotating members)

$$\left.\begin{aligned} &\phi_\ell = \frac{Ni}{\Re_\ell} \\ &\phi_m = \frac{Ni}{\Re_m} \end{aligned}\right\}$$

If the magnetic system is considered to be linear (saturation neglected), then, as in the case of stationary coupled circuits, we can express the fluxes in terms of reluctances.

$$\lambda = \left(\frac{N^2}{\Re_\ell} + \frac{N^2}{\Re_m} \right) i = \left(L_\ell + L_m \right) i$$

flux linkages

$L_\ell =$ leakage inductance

$L_m =$ magnetizing inductance

$$\Re_m = \Re_i + 2\Re_g$$ reluctance of the magnetizing path

$\Re_i$ $\begin{cases}\text{total reluctance of the magnetic material} \\ \text{of the stationary and movable members}\end{cases}$

$\Re_g$ reluctance of one of the air gaps

$$\left. \begin{aligned} \Re_i &= \frac{\ell_i}{\mu_{ri}\mu_0 A_i} \\ \Re_g &= \frac{x}{\mu_0 A_g} \end{aligned} \right\}$$

Assume that the cross-sectional areas of the stationary and movable members are equal and of the same material

$$A_g = A_i$$

This may be somewhat of an oversimplification, but it is sufficient for our purposes.

$$\Re_m = \Re_i + 2\Re_g$$

$$= \frac{1}{\mu_0 A_i}\left(\frac{\ell_i}{\mu_{ri}} + 2x\right)$$

$$L_m = \frac{N^2}{\frac{1}{\mu_0 A_i}\left(\frac{\ell_i}{\mu_{ri}} + 2x\right)}$$

Assume that the leakage inductance is constant.

The magnetizing inductance is clearly a function of displacement.

$x = x(t)$ and $L_m = L_m(x)$

When dealing with linear magnetic circuits wherein mechanical motion is not present, as in the case of a transformer, the change of flux linkages with respect to time was simply L(di/dt). This is not the case here.

$$\lambda(i,x) = L(x)i = [L_\ell + L_m(x)]i$$

$$\frac{d\lambda(i,x)}{dt} = \frac{\partial\lambda}{\partial i}\frac{di}{dt} + \frac{\partial\lambda}{\partial x}\frac{dx}{dt}$$

The inductance is a function of x(t).

$$v = ri + [L_\ell + L_m(x)]\frac{di}{dt} + i\frac{dL_m(x)}{dx}\frac{dx}{dt}$$

The voltage equation is a nonlinear differential equation.

$$L_m(x) = \frac{N^2}{\frac{1}{\mu_0 A_i}\left(\frac{\ell_i}{\mu_{ri}} + 2x\right)}$$

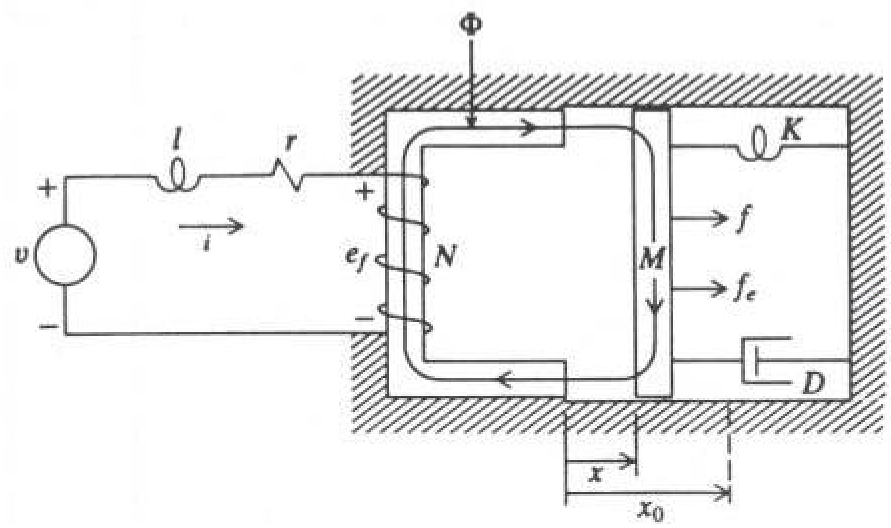
Let's look at the magnetizing inductance again.

$$L_m(x) = \frac{k}{k_0 + x} \qquad \begin{cases} k = \dfrac{N^2\mu_0 A_i}{2} \\[2ex] k_0 = \dfrac{\ell_i}{2\mu_{ri}} \end{cases}$$

$$L_m(0) = \frac{k}{k_0} = \frac{N^2\mu_0\mu_{ri}A_i}{\ell_i}$$

$$L_m(x) \cong \frac{k}{x} \quad \text{for } x > 0$$

Detailed diagram of electromagnet for further analysis



Electromagnet

$$L_m(x) \cong \frac{k}{x} \quad \text{for } x > 0$$

Use this approximation

$$L(x) \cong L_\ell + L_m(x) = L_\ell + \frac{k}{x} \quad \text{for } x > 0$$

$$\lambda(i,x) = L(x)i = [L_\ell + L_m(x)]i$$

The system is magnetically linear: $W_f(i,x) = W_c(i,x) = \frac{1}{2}L(x)i^2$

$$f_e(\vec{i},x) = \sum_{j=1}^{J}\left[i_j \frac{\partial \lambda_j(\vec{i},x)}{\partial x}\right] - \frac{\partial W_f(\vec{i},x)}{\partial x}$$

$$f_e(\vec{i},x) = \frac{\partial W_c(\vec{i},x)}{\partial x}$$

$$f_e(i,x) = \frac{1}{2}i^2 \frac{\partial L(x)}{\partial x} = -\frac{ki^2}{2x^2}$$

- The force f_e is always negative; it pulls the moving member to the stationary member. In other words, an electromagnetic force is set up so as to minimize the reluctance (maximize the inductance) of the magnetic system.
- Equations of motion:

$$v = ri + \ell \frac{di}{dt} + e_f$$

$$f = M\frac{d^2x}{dt^2} + D\frac{dx}{dt} + K\left(x - x_0\right) - f_e$$

Steady-State Operation
(if *v* and *f* are constant)

$$v = ri$$

$$f = K\left(x - x_0\right) - f_e$$

Steady-State Operation of an Electromagnet

$$f = K(x - x_0) - f_e$$

$$-f_e = f - K(x - x_0)$$

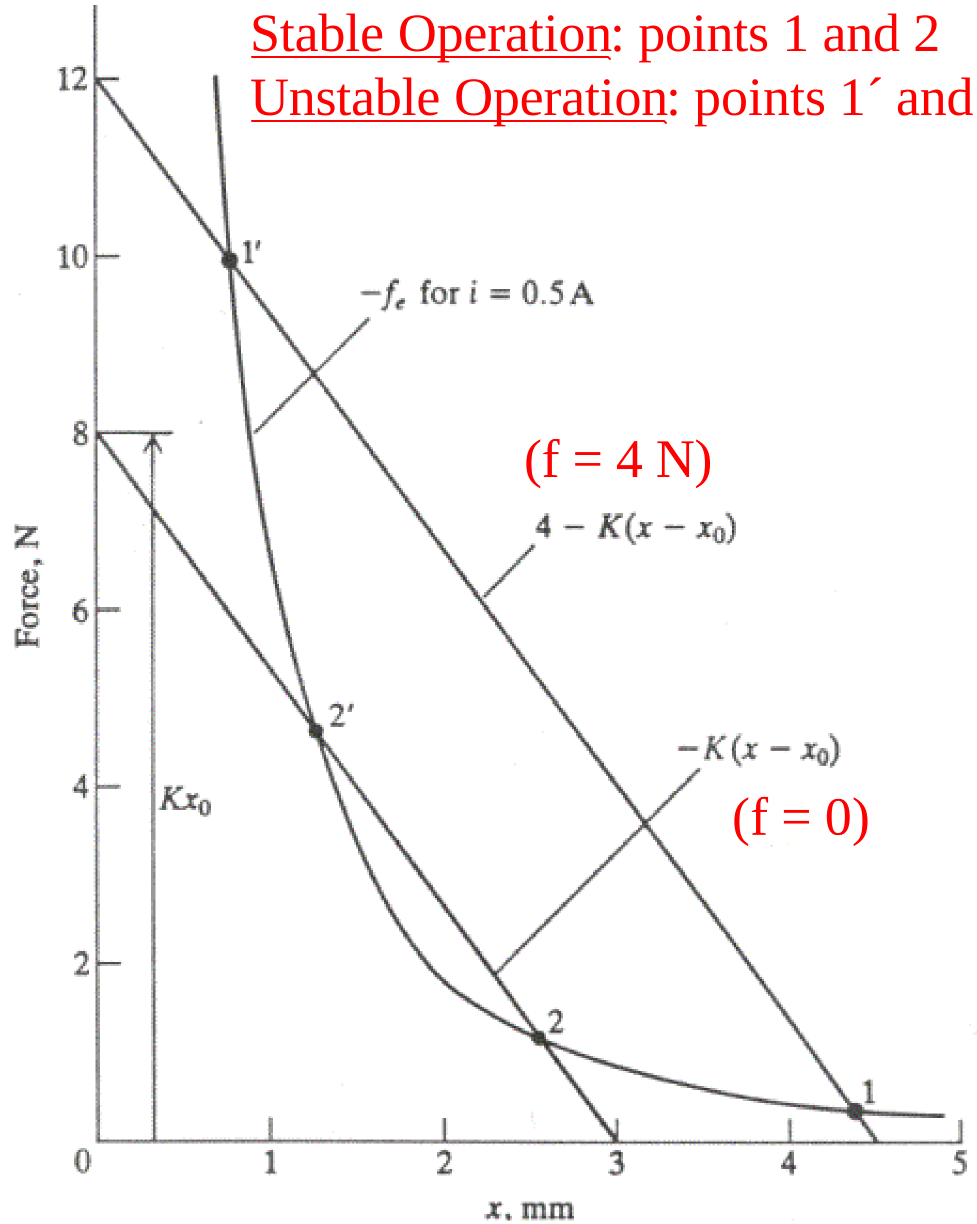
$$-\left(-\frac{ki^2}{2x^2}\right) = f - K(x - x_0)$$

Parameters:

- $r = 10\ \Omega$
- $K = 2667$ N/m
- $x_0 = 3$ mm
- $k = 6.283\text{E-}5$ H m
- $v = 5$ V
- $i = 0.5$ A

Stable Operation: points 1 and 2

Unstable Operation: points 1´ and 2´



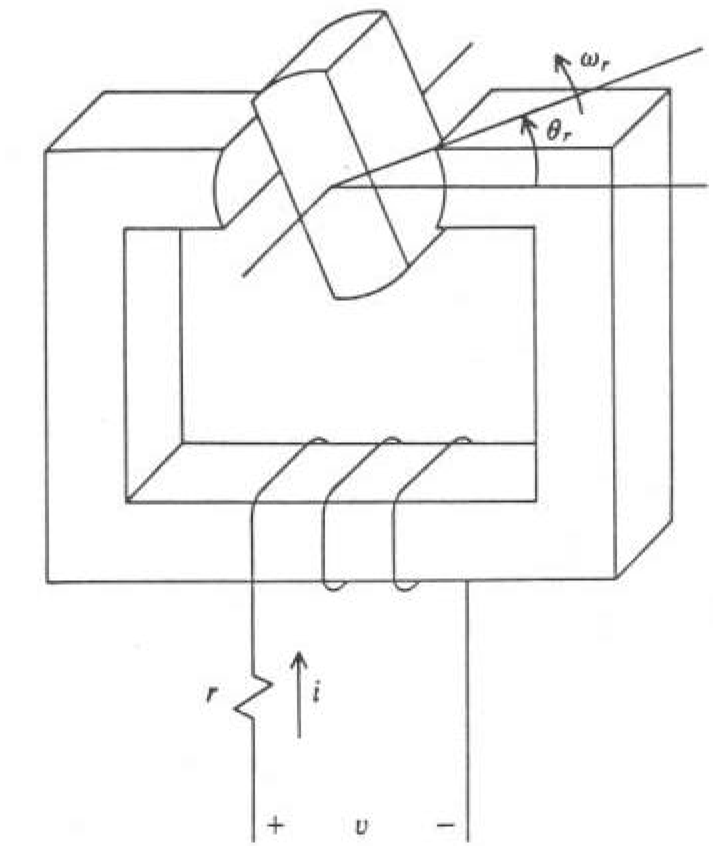
Single-Phase Reluctance Machine

- The machine consists of:
  - stationary core with a winding of N turns
  - moveable member which rotates

$\theta_r =$ angular displacement

$\omega_r =$ angular velocity

$$\theta_r = \int_0^t \omega_r(\xi)\,d\xi + \theta_r(0)$$



$$v = ri + \frac{d\lambda}{dt}$$

voltage equation

$$\phi = \phi_\ell + \phi_m$$

$\phi_\ell =$ leakage flux

$\phi_m =$ magnetizing flux

$$\lambda = \left(L_\ell + L_m\right) i$$

It is convenient to express the flux linkages as the product of the sum of the leakage inductance and the magnetizing inductance and the current in the winding.

$L_\ell =$ constant (independent of θ_r)

$L_m =$ periodic function of θ_r

$$L_m = L_m(\theta_r)$$

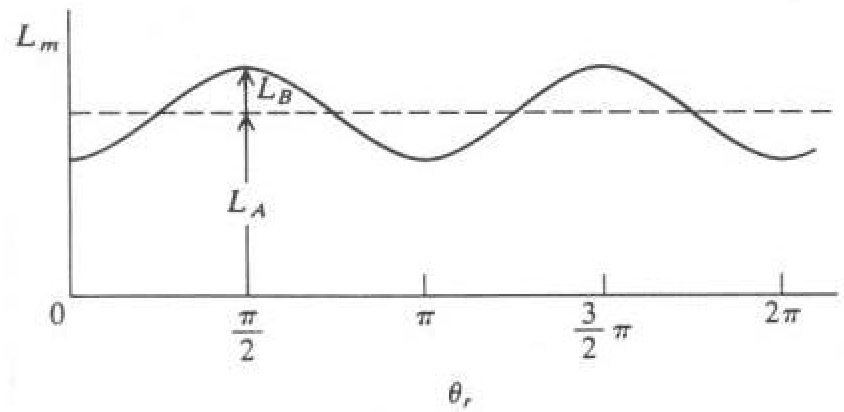
$$L_m(0) = \frac{N^2}{\Re_m(0)} \Rightarrow \left\{ \begin{array}{l} \Re_m \text{ is maximum} \\ L_m \text{ is minimum} \end{array} \right.$$

$$L_m\left(\frac{\pi}{2}\right) = \frac{N^2}{\Re_m\left(\frac{\pi}{2}\right)} \Rightarrow \left\{ \begin{array}{l} \Re_m \text{ is minimum} \\ L_m \text{ is maximum} \end{array} \right.$$

The magnetizing inductance varies between maximum and minimum positive values twice per revolution of the rotating member.

Assume that this variation may be adequately approximated by a sinusoidal function.

$$L_m(\theta_r) = L_A - L_B\cos(2\theta_r)$$



$$L_m(0) = L_A - L_B$$

$$L_m\left(\frac{\pi}{2}\right) = L_A + L_B$$

$$L_A > L_B$$

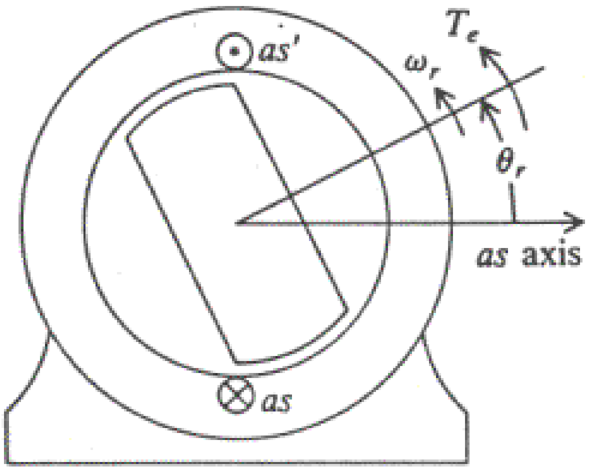
$$L_A = \text{ average value}$$

$$\begin{aligned} L(\theta_r) &= L_\ell + L_m(\theta_r) \\ &= L_\ell + L_A - L_B\cos(2\theta_r) \end{aligned}$$

$$v = ri + \left[L_\ell + L_m(\theta_r)\right]\frac{di}{dt} + i\,\frac{dL_m(\theta_r)}{d\theta_r}\frac{d\theta_r}{dt}$$

voltage equation

- This elementary two-pole single-phase reluctance machine is shown in a slightly different form. Winding 1 is now winding *as* and the stator has been changed to depict more accurately the configuration common for this device.



$$v_{as} = r_s i_{as} + \frac{d\lambda_{as}}{dt}$$

$$\lambda_{as} = L_{asas} i_{as}$$

$$L_{asas} = L_{\ell s} + L_A - L_B \cos(2\theta_r)$$

$$\theta_r = \int_0^t \omega_r(\xi) d\xi + \theta_r(0)$$

r_s = resistance of *as* winding

L_{asas} = self-inductance of *as* winding

$L_{\ell s}$ = leakage inductance

- Electromagnetic torque:
  - Magnetic system is linear, hence $W_f = W_c$.

$$W_c\left(i_{as},\theta_r\right)=\frac{1}{2}\left(L_{\ell s}+L_A-L_B\cos\left(2\theta_r\right)\right)i_{as}^2$$

$$T_e\left(\vec{i},\theta\right)=\sum_{j=1}^{J}\left[i_j\frac{\partial\lambda_j\left(\vec{i},\theta\right)}{\partial\theta}\right]-\frac{\partial W_f\left(\vec{i},\theta\right)}{\partial\theta}$$

$$T_e\left(\vec{i},\theta\right)=\frac{\partial W_c\left(\vec{i},\theta\right)}{\partial\theta}$$

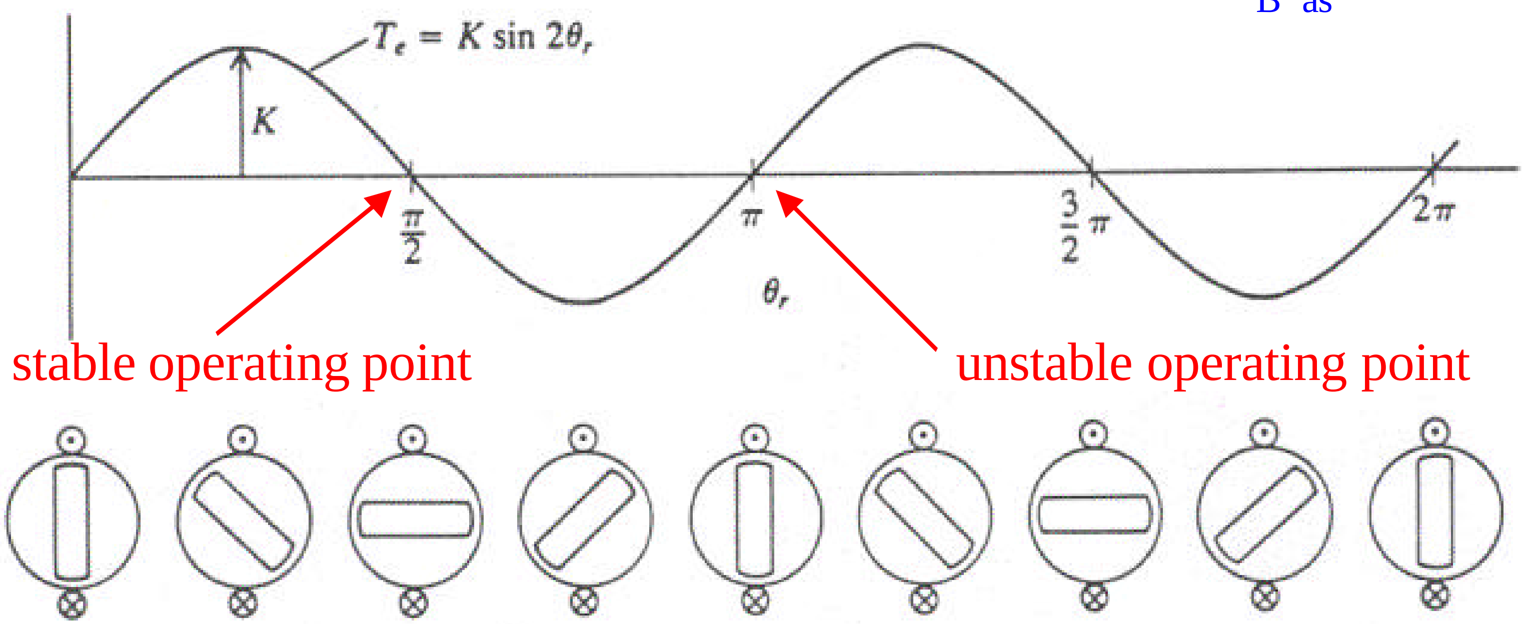
$$T_e\left(i_{as},\theta_r\right)=L_B i_{as}^2\sin\left(2\theta_r\right)$$

Valid for both transient and steady-state operation

- Consider steady-state operation: i_{as} is constant

$$T_e = K\sin\left(2\theta_r\right)$$

$$K = L_B i_{as}^2$$



Electromagnetic torque versus angular displacement of a single-phase reluctance machine with constant stator current

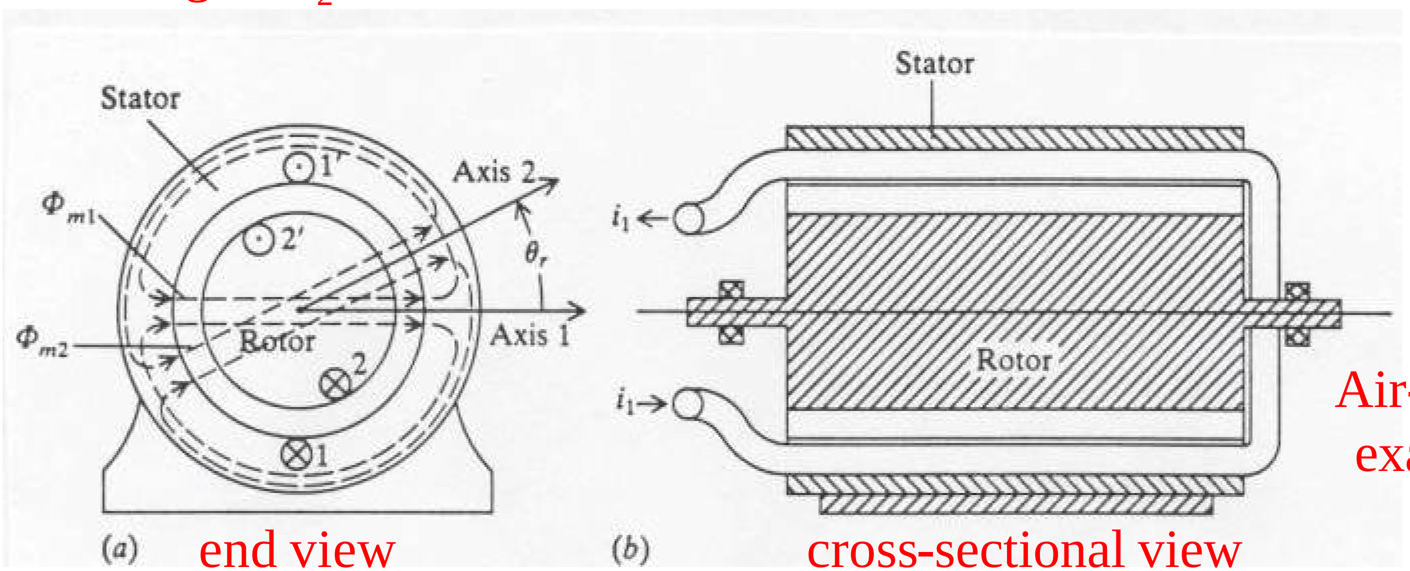
- Although the operation of a single-phase reluctance machine with a constant current is impracticable, it provides a basic understanding of reluctance torque, which is the operating principle of variable-reluctance stepper motors.
- In its simplest form, a variable-reluctance stepper motor consists of three cascaded, single-phase reluctance motors with rotors on a common shaft and arranged so that their minimum reluctance paths are displaced from each other.

Windings in Relative Motion

- The rotational device shown will be used to illustrate windings in relative motion.

Winding 1: N_1 turns on stator
Winding 2: N_2 turns on rotor

Assume that the turns are concentrated in one position.



end view

cross-sectional view

Air-gap size is exaggerated.

$$\boxed{\begin{aligned} v_1 &= r_1 i_1 + \frac{d\lambda_1}{dt} \\ v_2 &= r_2 i_2 + \frac{d\lambda_2}{dt} \end{aligned}}$$

voltage equations

$$\boxed{\begin{aligned} \lambda_1 &= L_{11} i_i + L_{12} i_2 \\ \lambda_2 &= L_{21} i_i + L_{22} i_2 \end{aligned}}$$

The magnetic system is assumed linear.

$$\boxed{\begin{aligned} L_{11} &= L_{\ell 1} + L_{m1} \\ &= \frac{N_1^2}{\Re_{\ell 1}} + \frac{N_1^2}{\Re_m} \\ L_{22} &= L_{\ell 2} + L_{m2} \\ &= \frac{N_2^2}{\Re_{\ell 2}} + \frac{N_2^2}{\Re_m} \end{aligned}}$$

The self-inductances L_{11} and L_{22} are constants and may be expressed in terms of leakage and magnetizing inductances.

$\Re_m$ is the reluctance of the complete magnetic path of φ_{m1} and φ_{m2} , which is through the rotor and stator iron and twice across the air gap.

Let's now consider L_{12}.

$$\theta_r = \text{ angular displacement}$$

$$\omega_r = \text{ angular velocity}$$

$$\theta_r = \int_0^t \omega_r(\xi)\, d\xi + \theta_r(0)$$

When θ_r is zero, then the coupling between windings 1 and 2 is maximum. The magnetic system of winding 1 aids that of winding 2 with positive currents assumed. Hence the mutual inductance is positive.

$$L_{12}(0) = \frac{N_1 N_2}{\Re_m}$$

When θ_r is $\pi/2$, the windings are orthogonal. The mutual coupling is zero.

$$L_{12}\left(\frac{\pi}{2}\right) = 0$$

Assume that the mutual inductance may be adequately predicted by:

$$\begin{cases} L_{12}(\theta_r) = L_{sr}\cos(\theta_r) \\ L_{sr} = \dfrac{N_1 N_2}{\Re_m} \end{cases}$$

$$\boxed{\begin{aligned} v_1 &= r_1 i_1 + \frac{d\lambda_1}{dt} \\ v_2 &= r_2 i_2 + \frac{d\lambda_2}{dt} \end{aligned}}$$

L_{sr} is the amplitude of the sinusoidal mutual inductance between the stator and rotor windings.

In writing the voltage equations, the total derivative of the flux linkages is required.

$$\lambda_1 = L_{11} i_1 + (L_{sr}\cos\theta_r) i_2$$

$$\lambda_2 = L_{22} i_2 + (L_{sr}\cos\theta_r) i_1$$

$$\boxed{\begin{aligned} v_1 &= r_1 i_1 + L_{11}\frac{di_1}{dt} + L_{sr}\cos\theta_r \frac{di_2}{dt} - i_2\omega_r L_{sr}\sin\theta_r \\ v_2 &= r_2 i_2 + L_{22}\frac{di_2}{dt} + L_{sr}\cos\theta_r \frac{di_1}{dt} - i_1\omega_r L_{sr}\sin\theta_r \end{aligned}}$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} L_{\ell 1} + L_{m1} & L_{sr} \cos\theta_r \\ L_{sr} \cos\theta_r & L_{\ell 2} + L_{m2} \end{bmatrix} \begin{bmatrix} i_{as} \\ i_{bs} \end{bmatrix}$$

Since the magnetic system is assumed to be linear:

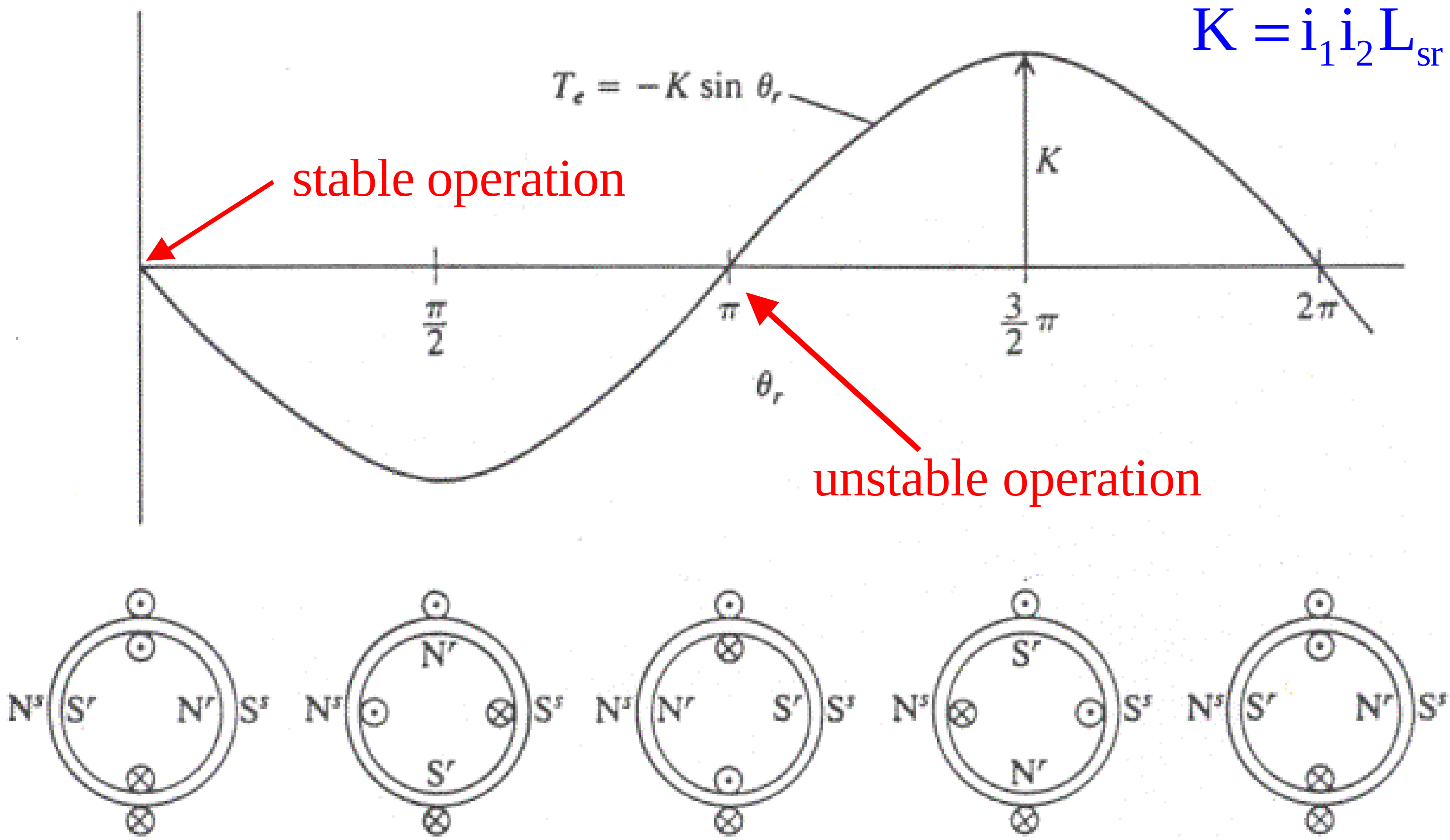
$$W_f\left(i_1, i_2, \theta_r\right) = \frac{1}{2} L_{11} i_1^2 + L_{12} i_1 i_2 + \frac{1}{2} L_{22} i_2^2 = W_c\left(i_1, i_2, \theta_r\right)$$

$$\left.\begin{aligned} T_e\left(\vec{i}, \theta\right) &= \sum_{j=1}^{J} \left[i_j \frac{\partial \lambda_j\left(\vec{i}, \theta\right)}{\partial \theta} \right] - \frac{\partial W_f\left(\vec{i}, \theta\right)}{\partial \theta} \\ T_e\left(\vec{i}, \theta\right) &= \frac{\partial W_c\left(\vec{i}, \theta\right)}{\partial \theta} \end{aligned}\right\} \quad T_e\left(i_1, i_2, \theta_r\right) = -i_1 i_2 L_{sr} \sin\theta_r$$

- Consider the case where i_1 and i_2 are both positive and constant:

$$T_e = -K \sin \theta_r$$

$$K = i_1 i_2 L_{sr}$$



Electromagnetic torque versus angular displacement with constant winding currents

- Although operation with constant winding currents is somewhat impracticable, it does illustrate the principle of positioning of stepper motors with a permanent-magnet rotor which, in many respects, is analogous to holding i_2 constant on the elementary device considered here.