SEMI-STATIC HEDGING OF BARRIER OPTIONS UNDER POISSON JUMPS

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We show that the payoff to barrier options can be replicated when the underlying price process is driven by the difference of two independent Poisson processes. The replicating strategy employs simple semi-static positions in co-terminal standard options. We note that classical dynamic replication using just the underlying asset and a riskless asset is not possible in this context. When the underlying of the barrier option has no carrying cost, we show that the same semi-static trading strategy continues to replicate even when the two jump arrival rates are generalized into positive even functions of distance to the barrier and when the clock speed is randomized into a positive continuous independent process. Since the even function and the positive process need no further specification, our replicating strategies are also semi-robust. Finally, we show that previous results obtained for continuous processes arise as limits of our analysis.

Keywords: Static replication; barrier options; hedging jump processes; robust valuation; Poisson processes; arbitrage.

1. Introduction

We suppose that one is interested in knowing how to replicate the payoff to a barrier option in a continuous time setting. In this paper, we will consider semi-static replication of this payoff which by definition implies that only a finite number of trades are performed after initiation. This kind of replication is to be contrasted with the more well known dynamic replication, which requires trading along the time continuum connecting trade initiation and the barrier option maturity.

For the types of barrier options we examine, our semi-static trading strategies require that at most one trade be performed after initiation. We don’t examine sequential barrier options, which might require more than one trade. The trade that might be made after initiation occurs at the first time before maturity, if any, that the price of the underlying asset hits or jumps over the barrier. At such a time, the strategy involves the buying and selling of European options in a self-financing way. The European options all mature at the same time $T$ as the barrier option. It turns out that no trades in the underlying asset or in the riskless asset are ever required.
We consider three different dynamical settings, increasing in both complexity and financial realism. The first setting is the simplest one, but has the financial drawbacks of requiring no carrying cost for the underlying asset and requiring symmetry in the risk-neutral price process. As prices are allowed to become arbitrarily large, the symmetry assumption leads to the possibility of negative price realizations for the underlying asset. The second setting is more complicated, but allows constant carrying costs and keeps prices positive, in order to be consistent with limited liability of the underlying asset. The third setting has the most complex dynamics, as it layers a possible jump to zero onto the second set of dynamics. We refer to the three settings as the arithmetic case, the geometric case, and the jump-to-default case respectively.

2. Arithmetic Case

2.1. Statistical and risk-neutral forward price process

In this section, we require that the underlying has no carrying cost. This arises if the barrier option is written on a forward price. It also arises if the barrier option is written on a spot price, but only under stringent conditions. For example, if the underlying asset is a stock, then the zero carry condition requires that the stock’s dividend yield happens to equal the riskfree rate. To cast the results of this section in their most favorable light, we will assume in this section that the barrier option is written on a forward price. The next section allows for nonzero carrying costs on the underlying asset.

Let $F_t$ be the forward price at time $t \in [0, T]$ for maturity $T' \geq T$. We assume that $F_t$ is a continuous time stochastic process. Let $\mathbb{P}$ denote statistical probability measure and let $N_1$ and $N_2$ denote two independent standard Poisson processes with positive jump arrival rates of $\alpha_1$ and $\alpha_2$. For some fixed positive constants $a$ and $F_0$, we assume that:

$$F_t = F_0 + a(N_1 t - N_2 t), \quad t \in [0, T].$$

(2.1)

In words, the forward price $F$ starts at $F_0 > 0$ and jumps up or down at independent exponential times by the amount $a$. Under $\mathbb{P}$, forward prices can change in expectation, due to a risk premium. Since the up and down jumps in the forward price have the same size, the existence of a nonzero risk premium can only be captured by a difference in the two jump arrival rates $\alpha_1$ and $\alpha_2$. In fact, the risk premium must have the same sign as $\alpha_1 - \alpha_2$. For any $t \in (0, T]$, the support of the probability mass function (PMF) of $F_t$ is the set $\{F_0 + az, z \in \mathbb{Z}\}$. As a result, the process has positive probability of becoming negative. We address this issue in the next section.

We note that the transition PMF of $F$ is known in closed form. To see this, we write $F_t = F_0 + aD_t$, where:

$$D_t \equiv N_1 t - N_2 t$$

(2.2)
is the difference of two standard Poisson processes. The PMF of a standard Poisson process \( N_t \) with intensity \( \alpha \) is of course:

\[
P\{N_t = n\} = \frac{e^{-\alpha t}(\alpha t)^n}{n!},
\]

for \( n \in \mathbb{N} \). The difference of two Poisson random variables is said to have a Skellam distribution. Extending the result of Skellam \[4\] in the obvious way to Poisson processes, we have that for \( k \in \mathbb{Z} \):

\[
P\{D_t = k\} = \sum_{n=-\infty}^{\infty} P\{N_{1t} = n + k\}P\{N_{2t} = n\}
= \sum_{n=-\infty}^{\infty} \frac{e^{-\alpha_1 t}(\alpha_1 t)^{n+k}}{(n+k)!} \frac{e^{-\alpha_2 t}(\alpha_2 t)^n}{n!},
\]

from (2.3), where any term with a negative factorial is zero. Factoring out the exponential functions:

\[
P\{D_t = k\} = e^{-(\alpha_1 + \alpha_2)t} \sum_{n=0}^{\infty} \frac{(\alpha_1 t)^{n+k}(\alpha_2 t)^n}{n!(n+k)!}
= e^{-(\alpha_1 + \alpha_2)t} \left( \frac{\alpha_1}{\alpha_2} \right)^{\frac{k}{2}} \sum_{n=0}^{\infty} \frac{(t\sqrt{\alpha_1 \alpha_2})^{2n+k}}{n!(n+k)!}
= e^{-(\alpha_1 + \alpha_2)t} \left( \frac{\alpha_1}{\alpha_2} \right)^{\frac{k}{2}} I_k(2t\sqrt{\alpha_1 \alpha_2}),
\]

since:

\[
I_k(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{2n+k}}{n!(n+k)!},
\]

defines the modified Bessel function.

We assume no arbitrage and hence the existence of a probability measure \( Q \) equivalent to \( P \) under which the forward price is a martingale. Let \( \lambda_1 \) and \( \lambda_2 \) denote the risk-neutral arrival rates of the two Poisson processes. The equivalence of \( Q \) and \( P \) implies that \( \lambda_1 \) and \( \lambda_2 \) are both positive. Imposing the martingale condition on \( F \) under \( Q \) implies that \( \lambda_1 = \lambda_2 \). Besides positivity, this is the only condition on the two intensities. As a result, the two intensities are not uniquely determined. If we let \( \lambda > 0 \) denote the common value of the two risk-neutral intensities. Then for each choice of \( \lambda \), there corresponds a risk-neutral measure \( Q \). The fact that \( \lambda \) can be any positive real number implies that there are an uncountably infinite number of martingale measures and that the market is incomplete. In particular, the payoff to an arbitrary barrier option cannot be replicated by dynamic trading in forward contracts and a riskfree asset.
From (2.5), the transition PMF of $F$ under $Q$ is also known in closed form since:

$$Q\{D_t = k\} = e^{-2\lambda t}I_k(2t\lambda),$$

for $k \in \mathbb{Z}$, which was first proved in Irwin [2]. The $F$ process is a conditionally symmetric $Q$ martingale and so by use of the reflection principle, the joint PMF of $F_T$ and its extrema is known in closed form. As a consequence, one can calculate a risk-neutral expectation of the payoff to a barrier option in this model. This expectation will depend on the common risk-neutral intensity $\lambda$. A sw ev a r y $\lambda$ over the interval $(0, \infty)$ while holding the riskfree rate and $F_0$ fixed, we generate a range of arbitrage-free values for the barrier option. The non-uniqueness in value is a consequence of the fact that the barrier option payoff cannot be replicated by dynamic trading in forward contracts and a riskless asset. In contrast, the next two subsections show that the barrier option payoff can be replicated by a simple semi-static position in co-terminal options. As a result, one obtains a unique arbitrage-free barrier option value, when the prices of these co-terminal options are taken as given.

2.2. Arithmetic put call symmetry

Let:

$$P_t(K) \equiv e^{-r(T-t)}E_t^Q(K - F_T)^+$$

and

$$C_t(K) \equiv e^{-r(T-t)}E_t^Q(F_T - K)^+$$

respectively denote the value at time $t \in [0, T]$ of a standard put and call with maturity $T$ and strike $K$. At any time $t \in [0, T]$, the conditional distribution of $F_T$ given $F_t$ is symmetric about $F_t$. As a consequence, put and call values are related by the following result termed Arithmetic Put Call Symmetry (APCS): For fixed underlying forward price $F_t$ and for any constant $c \in \mathbb{R}$:

$$P_t(F_t - c) = C_t(F_t + c).$$

**Proof.** To prove APCS, we evaluate (2.9) at $K = F_t + c$:

$$C_t(F_t + c) = e^{-r(T-t)}E_t^Q(F_T - (F_t + c))^+$$

$$= e^{-r(T-t)}E_t^Q(X_T - c)^+, \quad (2.11)$$

where $X_u \equiv F_u - F_t$ measures the distance of $F_u$ from $F_t$ for $u \geq t$. By the symmetry of the distribution of $X_T$ under $Q$, we have:

$$C_t(F_t + c) = e^{-r(T-t)}E_t^Q(-X_T - c)^+$$

$$= e^{-r(T-t)}E_t^Q(F_t - c - F_T)^+$$

$$= P_t(F_t - c). \quad (2.12)$$
2.3. Semi-static hedge of a one touch with payment at expiry

Without loss of generality, we focus on replicating the payoff of a one touch with payment at expiry. This is one of the most liquid kinds of barrier option in the FX markets at present. Without loss of generality, we suppose that we have a lower barrier \( L \in (0, F_0) \), which is active over the whole life \([0, T]\). The payoff on this one touch at \( T \) is \( 1_{\min_{t \in [0,T]} F_t \leq L} \). As the state space of the underlying forward price has been assumed to be discrete, it can matter for pricing that the inequality in the one touch payoff is weak, rather than strict.

It is clear from (2.1) that the process \( F_t - F_0 \) only takes values on the integers. The state space of the underlying forward price has been assumed to be discrete, it can matter for pricing that the inequality in the one touch payoff is weak, rather than strict.

Recall that for any real number \( r \), \( \lfloor r \rfloor \) denotes the largest integer weakly below it.

If the forward price \( F \) ever touches or crosses \( L \) from above, it lands at the level:
\[
L^a \equiv F_0 + a \left\lfloor \frac{L - F_0}{a} \right\rfloor, \tag{2.13}
\]
which is less than or equal to \( L \).

Let \( \tau \) be the first time in \([0,T]\), if any, that \( F \) crosses \( L \). At time \( \tau \), we have:
\[
F_\tau = L^a. \tag{2.14}
\]

Substituting (2.14) in APCS (2.10) and evaluating at \( c = \pm a \) implies:
\[
P_\tau (L^a - a) = C_\tau (L^a + a), \tag{2.15}
\]
and:
\[
P_\tau (L^a + a) = C_\tau (L^a - a). \tag{2.16}
\]

We consider a market in which one touches and standard puts and calls trade outright. Consider the payoff that arises from holding \( \frac{1}{a} \) vertical put spreads with strikes \( L^a \pm a \):
\[
\frac{1}{a} [(L^a + a - F_T)^+ - (L^a - a - F_T)^+] = 1_{(F_T < L^a)} + 0.5 \times 1_{(F_T = L^a)}. \tag{2.17}
\]
Likewise, the payoff that arises from holding \( \frac{1}{a} \) vertical call spreads with strikes \( L^a \pm a \) is:
\[
\frac{1}{a} [(F_T - (L^a - a))^+ - (F_T - (L^a + a))^+] = 1_{(F_T > L^a)} + 0.5 \times 1_{(F_T = L^a)}. \tag{2.18}
\]
We refer to the payoffs on the left-hand side of (2.17) and (2.18) as those of normalized put and call spreads respectively. We refer to the payoffs on the right-hand side of (2.17) and (2.18) as those of tiered binary puts and calls respectively. The semi-static hedge can be conducted using either normalized spreads or tiered binary options, which can obviously themselves be created out of non-tiered binary options. We will indicate the hedge using just normalized spreads, as the standard options used to construct them are more liquid in practice.
To hedge the initial sale of a one touch, consider buying two normalized put
spreads, as defined above. If the forward price \( F \) stays above the lower barrier \( L \)
over the time interval \([0,T]\), then the put spreads expire worthless, as does the
one touch. On the other hand, if \( F \) touches or crosses \( L \) at least once in \([0,T]\),
then at the first such time, one of the normalized put spreads in inventory can be
immediately sold. In particular, the long position in \( \frac{1}{a} \) puts struck at \( L^a + a \) can be
sold and from (2.16), the money generated is just sufficient to buy \( \frac{1}{a} \) calls struck at
\( L^a - a \). Similarly, the short position in \( \frac{1}{a} \) puts struck at \( L^a - a \) can be covered and
from (2.15), the money needed to cover this short position is generated by writing \( \frac{1}{a} \)
calls struck at \( L^a + a \). If these self-financing trades are made, then the hedger ends
up owning a normalized call spread and a normalized put spread, whose combined
payoff is one dollar regardless of the terminal state. This total receipt of one dollar
is just sufficient to meet the liability due to the one touch finishing in-the-money.
Thus, the payoff of the one touch can be exactly replicated by this simple semi-static
trading strategy, whether or not the barrier is ever touched.

Let \( OTPE_0(L) \) be the initial price of a one touch with payment at its maturity
\( T \) and with lower barrier \( L \). Recall that \( P_0(K) \) denotes the initial price of a standard
put with maturity \( T \) and strike \( K \). The only arbitrage-free price of the one touch
is the cost of setting up the semi-static hedge:

\[
OTPE_0(L) = \frac{2}{a} [P_0(L^a + a) - P_0(L^a - a)],
\]

where recall:

\[
L^a = F_0 + a \left[ \frac{L - F_0}{a} \right].
\]

In words, the one touch has the same value as two normalized put spreads, cen-
tered on the landing level \( L^a \). Each put spread is said to have width \( a \) and by the
normalization is said to have height one. We note that normalized put spreads with
other widths can be used in the semi-static hedge. In particular, a one touch is also
hedged by initially holding two normalized put spreads centered at \( L^a \) and with
width \( w \in (0,a) \), so that:

\[
OTPE_0(L) = \frac{2}{w} [P_0(L^a + w) - P_0(L^a - w)].
\]

This flexibility can be useful when there are constraints on the strikes that trade.

We note that given the market prices of the two puts in the hedge, the risk-
neutral arrival rate \( \lambda \) does not enter the valuation formula (2.19). It follows that
the same semi-static hedge succeeds if the common risk-neutral arrival rate is gen-
eralized into a deterministic function of time \( \lambda(t) \). In fact, by allowing this function
to be a superposition of delta functions, i.e.

\[
\lambda(t) = \sum_{n=0}^{\infty} \delta(t - n\Delta t)
\]

for some given positive constant \( \Delta t \), we obtain a discrete-time trinomial process.
Since the deterministic function of time does not need to be known for the same semi-static hedge to succeed, it follows that the two Poisson processes can be generalized into Cox processes, provided that the required trade at the first barrier crossing time remains self-financing. In turn, the self-financing requirement hinges on the symmetry about \( L^a \) of the PMF of \( F_T \), once one conditions on the first time, if any, that \( F \) has touched or crossed \( L \). In other words, the same semi-static hedge will succeed provided that the stochastic process governing the common risk-neutral jump arrival rate is consistent with this symmetry condition. A sufficient condition for this process is that it evolves independently of the two driving Poisson processes. The process does not need to be specified further. As a result, we say that the semi-static hedge is semi-robust.

Even more general dynamics are possible if the risk-neutral dynamics of the forward price \( F \) are allowed to depend on the location \( L \) of the lower barrier. While this may seem peculiar at first glance, we note that historically, the level at which the barrier of a barrier option has been placed has been precisely where the dynamics of the underlying were thought to change should the barrier be crossed. For example, when the European Exchange Rate Mechanism (ERM) was active, the barriers of double knockouts were frequently chosen to be the levels which central banks of participating sovereign entities were obliged to defend.

To generalize the risk-neutral dynamics of \( F \) when these dynamics can depend on \( L \), we re-interpret the risk-neutral forward price process as a compound Poisson process, where the intensity of the driving Poisson process is \( 2\lambda \), and the jump size distribution is symmetric Bernoulli. Under this interpretation, the jump size realizes to \( \pm a \) with equal probability of one half. Let \( \tau \equiv \inf_{t \in [0,T]} F_t \leq L \) be the first passage time of \( F \) to the barrier \( L \). For \( t \in [0,\tau] \), the intensity of the driving Poisson process can be any stochastic process, including those whose dynamics depend on the forward price or additional factors. For \( t \in [\tau,T] \), a sufficient condition on the forward price dynamics which allows the same static hedge to succeed is that the forward price is a compound Poisson martingale, whose jump size distribution is symmetric, but not necessarily Bernoulli. The arrival rate for each possible jump size of absolute value \( a > 0 \) is:

\[
\lambda_t(a) = e(F_t - L^a; a)\alpha_t(a), \quad t \in [\tau,T],
\]

where for each fixed \( a \), \( e(x;a) : \mathbb{R} \to \mathbb{R}^+ \) is an even nonnegative function and \( \alpha \) is a nonnegative stochastic process that evolves independently of the driving Poisson process. In words, the arrival rate for each absolute jump size can depend on the level of the process after the barrier crossing time, so long as this dependence is even in the distance from the landing level \( L^a \). The arrival rate for each absolute jump size can depend on an arbitrary number of additional factors after the barrier crossing time, so long as these factors evolve independently of the driving Poisson process. The function \( e \) and the process \( \alpha \) do not need to be specified further, so the semi-static hedge is semi-robust.
Instead of generalizing the forward price dynamics, we can instead consider a special case which arises as a limit. Recall that the forward price $F$ is a martingale with mean $F_0$. Its quadratic variation is stochastic:

$$[F, F]_t = a^2(N_{1t} + N_{2t}), \quad t \in [0, T],$$

(2.23)

while its variance grows linearly with time:

$$\text{Var}(F_t) = E[F, F]_t = a^2 2\lambda t, \quad t \in [0, T].$$

(2.24)

Suppose we now set the jump amplitude $a = \frac{\sigma}{\sqrt{2\lambda}}$, so that for any $\lambda$, we have:

$$\text{Var}(F_t) = \sigma^2 t, \quad t \in [0, T].$$

(2.25)

As $\lambda$ increases to infinity, the absolute jump size $a$ decreases to zero, and both the $F$ process and its quadratic variation become continuous. As $\lambda$ increases to infinity, the number of jumps along any path becomes infinite almost surely and by the law of large numbers, the quadratic variation of $F$ approaches its mean, i.e. the variance of $F$, which is proportional to time. Hence, by Lévy’s characterization of Brownian motion, there exists a standard Brownian motion $W$ such that in the limit:

$$\lim_{\lambda \to \infty} F_t = F_0 + \sigma W_t, \quad t \in [0, T].$$

(2.26)

We also get from (2.20) that:

$$\lim_{\lambda \to \infty} L^a = \lim_{a \downarrow 0} L^a = L,$$

(2.27)

and hence from (2.19) that:

$$\lim_{\lambda \to \infty} \text{OTPE}_0(L) = 2P_0'(L).$$

(2.28)

Thus, these well-known results concerning semi-static hedging in the Bachelier model arise as special cases of our analysis. We remark that the payoff of the one-touch can be replicated by dynamic trading in a forward contract and the riskfree asset in the limit. The option hedge in (2.19) also uses two assets, but differs in that there is at most one trade after initiation rather than an uncountably infinite number. We repeat that the semi-static option hedge works when $\lambda$ is finite, while replication even with fully dynamic trading in the underlying and the riskfree asset is not possible for any finite $\lambda$.

The next section deals with the complications that arise if we allow for carrying costs on the underlying and if we further require that the underlying price process stays positive.

3. Geometric Case

3.1. Statistical and risk-neutral spot price process

In this section, we will assume that the barrier option is written on the spot price of some underlying asset. To construct the process describing the underlying spot
price dynamics under the statistical probability measure $\mathbb{P}$, we again let $N_1$ and $N_2$ denote two independent standard Poisson processes, with positive jump arrival rates $\alpha_1$ and $\alpha_2$ respectively. For given positive constants $g$ and $S_0$, we assume that the stochastic process governing the spot price of the underlying asset is given by:

$$S_t = S_0 e^{g(N_1 - N_2)}, \quad t \in [0, T].$$

(3.1)

In words, the spot price $S$ starts at $S_0 > 0$ and jumps up by the amount $S_t - (e^g - 1) > 0$ or down by the amount $S_t - (e^{-g} - 1) < 0$ at independent exponential times. Since the spot price is the exponential of a Lévy process, it is always positive. The convexity of the exponential function implies that the up jump in $S$ always has a bigger absolute amplitude than the down jump in $S$, since $|e^g - 1| > |e^{-g} - 1|$. Let $r \in \mathbb{R}$ be the assumed constant riskfree rate, $q \in \mathbb{R}$ be the assumed constant dividend yield, and $\pi \in \mathbb{R}$ be the assumed constant risk premium. Then the two jump arrival rates $\alpha_1$ and $\alpha_2$ must produce an expected exponential growth rate of $r - q + \pi$ per unit time:

$$\alpha_1(e^g - 1) + \alpha_2(e^{-g} - 1) = r - q + \pi.$$

(3.2)

We assume no arbitrage and hence the existence of a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which the forward price of the underlying asset is a martingale. We will sometimes refer to any martingale measure as a risk-neutral measure. Let $\lambda_1$ and $\lambda_2$ denote the risk-neutral arrival rates of the two Poisson processes. The equivalence of $\mathbb{Q}$ and $\mathbb{P}$ implies that $\lambda_1$ and $\lambda_2$ are both positive. Under a risk-neutral measure $\mathbb{Q}$, the spot price has an expected exponential growth rate of $r - q$ per unit time and imposing this condition implies that:

$$\lambda_1(e^g - 1) + \lambda_2(e^{-g} - 1) = r - q.$$

(3.3)

Solving for $\lambda_2$ implies:

$$\lambda_2 = \lambda_1 e^g - \frac{r - q}{1 - e^{-g}}.$$

(3.4)

For future use, we note that $\lambda_2 \geq \lambda_1 e^g$ if and only if $r \leq q$. When $r = q$, (3.4) shows that the ratio of the two intensities is fixed:

$$\frac{\lambda_2}{\lambda_1} = e^g > 1.$$

(3.5)

Thus, when $r = q$, the spot price $S$ is a martingale and $\lambda_2$ must be larger than $\lambda_1$ by a factor of $e^g$ because the down jump size is smaller in absolute value than the up jump size by a factor of $e^{-g}$. When $r \neq q$, then we see from (3.4) that for $r > q$, $\lambda_2$ does not need to be as large, since the spot price has positive drift. However, we do require that $\lambda_2$ be positive. A simple sufficient condition for this is $r \leq q$.

\footnote{In the FX context, $q$ denotes the foreign riskfree rate. In the commodities context, $q$ denotes the convenience yield.}
If \( r > q \), then we require that \( r \) and/or \( q \) be constrained so that the growth rate in the underlying spot price is bounded above:

\[
r - q < \lambda_1(e^g - 1).
\]

This condition arises due to our desire to keep the set of possible spot prices be lines parallel to the (flat) barrier of the barrier option. Later, we will examine the implications of making the arrival rates a function of \( g \) and letting \( g \downarrow 0 \). We will find that the upper bound on \( r - q \) is eliminated.

Besides requiring positivity of \( \lambda_1 \) and \( \lambda_2 \), (3.3) is the only condition on the two intensities. As a result, the two intensities are not uniquely determined by knowledge of just \( g, r, \) and \( q \). For every ordered pair \((\lambda_1, \lambda_2)\) satisfying (3.3) and positivity, there corresponds a risk-neutral measure \( Q \). The fact that there are a continuum of positive solutions to (3.3) again implies that there are an uncountably infinite number of martingale measures and that the market is incomplete.

Let:

\[
Y_t \equiv \ln(S_t/S_0) = g(N_{1t} - N_{2t}), \quad t \in [0, T],
\]

be a measure of the distance from the starting point. \( Y \) is a pure jump Lévy process with starting value zero and under \( Q \), the Lévy measure of \( Y \) is:

\[
k_Q(dy) = [\lambda_1 \delta(y-g) + \lambda_2 \delta(y+g)]dy.
\]

Since \( \lambda_1 < \lambda_2 \), \( Y \) is not a \( Q \) martingale. However, an Esscher transform (see Shiryaev [3] p. 683) can be used to construct a new probability measure equivalent to \( Q \) under which \( Y \) is a pure jump Lévy martingale. We begin by noting that for any real constant \( c \):

\[
M_t(c) \equiv e^{cy_t - t[\lambda_1(e^{cg} - 1) + \lambda_2(e^{-cg} - 1)]}, \quad t \in [0, T],
\]

is a family of positive martingales started at one.

Define a new probability measure \( Q(c) \) by:

\[
dQ(c) = M_T(c)dQ.
\]

Under \( Q(c) \), the Lévy measure of \( Y \) is:

\[
k_{Q(c)}(dy) = e^{cy}[\lambda_1 \delta(y-g) + \lambda_2 \delta(y+g)]dy
\]

\[
= [e^{cg}\lambda_1 \delta(y-g) + e^{-cg}\lambda_2 \delta(y+g)]dy.
\]

Under \( Q(c) \), \( Y \) is the difference of two Poisson processes with intensities \( e^{cg}\lambda_1 \) and \( e^{-cg}\lambda_2 \) respectively. As we raise \( c \), we raise the first intensity and lower the second one. Let \( p \) denote the specific value of the parameter \( c \) that equates the two intensities:

\[
e^{pg}\lambda_1 = e^{-pg}\lambda_2.
\]
Solving for $p$:

$$p = \frac{\ln(\lambda_2/\lambda_1)}{2g}.$$  \hfill (3.11)

Hence $N_1$ has the same intensity under $Q(p)$ as $N_2$.

Substituting (3.4) in (3.11) eliminates the dependence of $p$ on $\lambda_2$:

$$p = \frac{1}{2} + \ln \left(1 - \frac{r - q}{\lambda_1(e^g - 1)}\right).$$  \hfill (3.12)

When $r \neq q$, $p$ depends on $\lambda_1$, but when $r = q$, $p = 1/2$ irrespective of the values taken by the two intensities.

In the next two subsections, we will show that the positions taken in the hedge instruments depend on $p$. Since $p$ depends on the risk-neutral intensity $\lambda_1$ when $r \neq q$, $\lambda_1$ must be known in this case. To infer it, we now assume that a variance swap trades at time 0. This swap pays $\frac{V_T^2}{g} - V_0^2(T')$ at time $T'$, where the variance swap rate $V_0$ is determined so that the contract has zero cost of entry at time 0. It is straightforward to show that $V_0$ is independent of $T'$ and is given in our model by:

$$V_0 = g\sqrt{\lambda_1 + \lambda_2}.$$  \hfill (3.13)

Substituting (3.4) in (3.13) shows that $\lambda_1$ is determined:

$$\lambda_1 = \frac{1}{1 + e^g} \left[ \frac{V_0^2}{g^2} + \frac{r - q}{1 - e^{-g}} \right].$$  \hfill (3.14)

Substituting (3.14) in (3.4) shows that $\lambda_2$ is also determined:

$$\lambda_2 = \frac{1}{1 + e^g} \left[ \frac{V_0^2}{g^2} + e^g(r - q) \right].$$  \hfill (3.15)

On the other hand, if $r = q$, or if the barrier option is written on a forward price, then the positions taken in the hedge instruments do not depend on $\lambda_1$ and $\lambda_2$. As a result, one does not need to know the price of a variance swap or any other asset (including the underlying stock and bond) to determine the positions taken in the hedge instruments. This allows the same hedge to succeed under more general conditions, whose precise specification does not need to be known.

### 3.2. Geometric put call symmetry

For a fixed $t \in [0, T]$, we may define a process $X_u \equiv \ln(S_u/S_t)$ on the time interval $u \in [t, T]$. At any time $t \in [0, T]$, the distribution under $Q$ of $X_T \equiv \ln(S_T/S_t)$ given $S_t$ is not symmetric. However, if we change numeraires appropriately, then this distribution is symmetric about zero. It will follow that a call has the same value as a portfolio of puts. If $r = q$, then we will show that this portfolio reduces to a single put with an appropriately chosen strike. Hence when $r = q$, puts and calls of equal moneyness have the same value once their payoffs are properly scaled. As a consequence, we refer to this ensemble of results as Geometric Put Call Symmetry (GPCS).
To state GPCS formally, let:
\[ C_t(K) = e^{-r(T-t)} E_t^Q (S_T - K)^+ \]  
(3.16)
denote the value at time \( t \in [0, T] \) of a standard call with maturity \( T \) and strike \( K \). When the risk-neutral spot price dynamics are as given in (3.1) and (3.3), then for any \( \ell \in \mathbb{R} \):
\[ C_t(S_t e^\ell) = e^\ell e^{-r(T-t)} E_t^Q \left( \frac{S_T}{S_t} \right)^{\frac{1}{g} \ln \left( \frac{\lambda_2}{\lambda_1 e^g} \right)} (S_t e^{-\ell} - S_T)^+. \]  
(3.17)
In words, (GPCS) states that when the spot price is at \( S_t \), a call struck at \( S_t e^\ell \) has \( e^\ell \) times the value of a claim paying \( \left( \frac{S_T}{S_t} \right)^{\frac{1}{g} \ln \left( \frac{\lambda_2}{\lambda_1 e^g} \right)} (S_t e^{-\ell} - S_T)^+ \) dollars at \( T \). The payoff of the latter claim is path-independent, continuous, and vanishes for \( S_T > S_t e^\ell \). As a consequence, its payoff is spanned by the payoff from a portfolio of puts struck at \( S_t e^{-\ell} \) and below.

**Proof.** To prove GPCS, we evaluate (3.16) at \( K = S_t e^\ell \):
\[ C_t(S_t e^\ell) = e^{-r(T-t)} E_t^Q (S_T - S_t e^\ell)^+ \]
\[ = e^{-r(T-t)} E_t^Q (S_t e^{X_T} - S_t e^\ell)^+ \]
\[ = e^{-r(T-t)} S_t E_t^Q e^{p X_T - \psi(p)(T-t)} (e^{(1-p) X_T} - e^{\ell - p X_T})^+ e^{\psi(p)(T-t)}, \]  
(3.18)
from algebra. If \( \psi(p) = \lambda_1 (e^{pg} - 1) + \lambda_2 (e^{-pg} - 1) \), then \( e^{p X_T - \psi(p)(T-t)} = \frac{M_T(p)}{M_T(p)} \), which can be used to Esscher transform the measure to \( Q(p) \):
\[ C_t(S_t e^\ell) = e^{-r(T-t)} S_t E_t^Q (e^{(1-p) X_T} - e^{\ell - p X_T})^+ e^{\psi(p)(T-t)} \]
\[ = e^{-r(T-t)} S_t E_t^Q (e^{(p-1) X_T} - e^{p X_T})^+ e^{\psi(p)(T-t)}, \]  
(3.19)
by the symmetry of the distribution of \( X_T \) under \( Q(p) \). Esscher transforming the measure back to \( Q \):
\[ C_t(S_t e^\ell) = e^{-r(T-t)} S_t E_t^Q e^{p X_T - \psi(p)(T-t)} (e^{(p-1) X_T} - e^{\ell + p X_T})^+ e^{\psi(p)(T-t)} \]
\[ = e^{-r(T-t)} E_t^Q e^{(2p-1) X_T + \ell} (S_t e^{-\ell} - S_t e^{X_T})^+ \]
\[ = e^\ell e^{-r(T-t)} E_t^Q \left( \frac{S_T}{S_t} \right)^{2p-1} (S_t e^{-\ell} - S_T)^+. \]  
(3.20)
But from (3.11):
\[ 2p - 1 = \frac{1}{g} \ln \left( \frac{\lambda_2}{\lambda_1 e^g} \right), \]  
(3.21)
and substituting (3.21) in (3.20) produces the desired result. \( \square \)

Let:
\[ P_t(K) = e^{-r(T-t)} E_t^Q (K - S_T)^+ \]  
(3.22)
denote the value at time $t \in [0, T]$ of a standard put with maturity $T$ and strike $K$. The appendix proves that the put portfolio with the same value as the call is given by:

$$C_t(S_t e^\ell) = e^{2\ell(1-p)} P_t(S_t e^{-\ell}) + 2(2p - 1) \times \int_0^{S_t e^{-\ell}} \left\{ \left( \frac{K}{S_t} \right)^{2p-2} \left[ \frac{p-1}{K} + \frac{pe^\ell}{S_t} \right] \right\} P_t(K) dK,$$

(3.23)

where $p$ is given in (3.12).

GPCS simplifies considerably when the underlying has no carrying cost, e.g. when the underlying is a forward price or if the underlying is a spot price and $r = q$. When $r = q$, then $\lambda_2 = \lambda_1 e^\ell$ from (3.4), $p = \frac{1}{2}$ from (3.12), and hence (3.23) simplifies to:

$$C_t(S_t e^\ell) = e^\ell P_t(S_t e^{-\ell}),$$

(3.24)

for any constant $\ell \in \mathbb{R}$. Suppose that we say that two options have equal moneyness if the geometric mean of their strikes is the current forward price. For example, a call struck at 110 and a put struck at 90.91 are both 10% out-of-the-money if the forward is at 100. When $r = q$, (3.24) states that the call that is $\ell$% out-of-the-money has $\ell$% more value than the put that is $\ell$% out-of-the-money.

To obtain the intuition for this result, suppose we consider the payoff from a call struck at $S_t e^\ell$. Instead of considering this payoff as a function of $S_T$ when pure discount bonds are the numeraire, we may consider this payoff as a function of $X_T$ when a claim paying $\sqrt{S_T S_t}$ is the numeraire. This payoff has an odd extension which turns out to be that of a short position in $e^\ell$ puts struck at $S_t e^{-\ell}$. Since the terminal distribution of $X_T$ is even under the measure $\mathbb{Q}(1/2)$ associated with the root claim as numeraire, the value of the odd payoff is zero. Geometric Put Call Symmetry follows as a result.

### 3.3. Semi-static hedge of a one touch with payment at expiry

We again consider the problem of hedging the sale of a one touch with lower barrier $L < S_0$. If the spot price $S$ ever touches or crosses $L$ from above, it lands at the level:

$$L^g \equiv S_0 e^{\ell \left\lfloor \frac{\ln(S_t / S_0)}{\tau} \right\rfloor},$$

(3.25)

which is less than or equal to $L$. Let $\tau$ be the first time in $[0, T]$ that $S$ touches or crosses $L$. We have that $S_\tau = L^g$. Evaluating geometric put call symmetry at time $\tau$, we have that:

$$C_\tau(L^g e^\ell) = e^\ell e^{-r(T-\tau)} E^\mathbb{Q}_\tau \left( \frac{S_T}{L^g} \right)^{\frac{1}{2} \left( \frac{\lambda_2}{\lambda_1} \right)} (L^g e^{-\ell} - S_T)^+.$$

(3.26)
Let:

\[ X_t \equiv \ln\left( \frac{S_t}{L^g} \right) = \ln\left( \frac{S_0}{L^g} \right) + g(N_1 - N_2), \quad t \in [0, T], \tag{3.27} \]

be a measure of the distance to the landing level. Comparing (3.6) and (3.27) implies that \( X \) differs from \( Y \) only in its starting value. Hence, \( X \) is a pure jump Lévy process with positive starting value \( \ln\left( \frac{S_0}{L^g} \right) \) and under \( Q \), the Lévy measure of \( X \) is:

\[ k_Q(dx) = [\lambda_1 \delta(x - g) + \lambda_2 \delta(x + g)]dx. \]

Since the arrival rate of up jumps is smaller than the arrival rate of down jumps, \( X \) is not a \( Q \) martingale. However, if we switch the measure to \( Q(p) \), then up jumps in \( X \) have the same arrival rate as down jumps. As a consequence, \( X \) is a \( Q(p) \) martingale.

To develop the hedge for the sale of a one touch, we recall that in the arithmetic case, the hedge had a purely static component and a semi-static component. In that setting, both components were given by a normalized put spread. If the barrier was crossed, the normalized put spread representing the semi-static component was flipped into a normalized call spread. In the present geometric setting, we will continue to use a normalized put spread for the fully static component. The semi-static component will not be the same normalized put spread, but rather a European-style payoff that can be flipped into a normalized call spread.

We let:

\[ K_\ell \equiv L^g e^{-g} \tag{3.28} \]

be the lower strike in the normalized put spread and we let:

\[ K_h \equiv L^g e^g \tag{3.29} \]

be the higher strike. We note that \( L^g \) is the geometric average of the two strikes:

\[ L^g = \sqrt{K_\ell K_h}. \tag{3.30} \]

The purely static component of the hedge again consists of \( \frac{1}{K_h - K_\ell} \) vertical put spreads with strikes \( K_\ell \) and \( K_h \). The payoff from this put spread is:

\[ \frac{1}{K_h - K_\ell} [(K_h - S_T)^+ - (K_\ell - S_T)^+]. \tag{3.31} \]

This payoff is constant at unity for \( S_T \leq K_\ell \), a declining linear function for \( S_T \in (K_\ell, K_h) \), and zero for \( S_T \geq K_h \). In particular, when \( S_T = L^g \), then the payoff is \( e^{\frac{g}{1 - \alpha}} - 1 \).

The semi-static component of the hedge is a portfolio of puts whose combined payoff is:

\[ = \frac{1}{K_h - K_\ell} \left( \frac{S_T}{L^g} \right)^{\frac{1}{2} \ln\left( \frac{S_T}{L^g} \right)} [e^{-g}(K_h - S_T)^+ - e^g(K_\ell - S_T)^+], \tag{3.32} \]
This payoff vanishes for $S_T \geq K_h \equiv L^g e^g$. The normalized put spread payoff in (3.31) also vanishes for $S_T \geq K_h \equiv L^g e^g$. Thus, the sum of the two payoffs is zero if $S$ never touches or crosses $L$.

If $S$ does touch or cross $L$, then from applying GPCS (3.26) at both $\ell = g$ and $\ell = -g$, it is easily shown that the payoff in (3.32) has the same value at the crossing time $\tau$ as a normalized call spread. Hence one can liquidate the put portfolio and buy a position in $\frac{1}{K_h - K_\ell}$ vertical call spreads with strikes $K_\ell$ and $K_h$:

$$\frac{1}{K_h - K_\ell} [(S_T - K_\ell)^+ - (S_T - K_h)^+].$$

(3.33)

This is the payoff of a normalized call spread. This payoff combines with the normalized put spread payoff in (3.31) to produce a constant payoff of one dollar at $T$. In particular, when $S_T = L^g$, the then payoff from the normalized call spread is $\frac{1 - e^{-g}}{e^{-g} - 1}$. Recall that when $S_T = L^g$, then the payoff from the normalized put spread is $\frac{e^g - 1}{e^g - 1}$. Thus, when $S_T = L^g$, the sum of the two payoffs is unity as required.

Summing the payoffs in (3.31) and (3.32), we conclude that the sale of a one touch is hedged by holding a portfolio of co-terminal European options paying:

$$\frac{1}{K_h - K_\ell} \left\{ (K_h - S_T)^+ - (K_\ell - S_T)^+ \right\} + \left( \frac{S_T}{L^g} \right)^{\frac{1}{2}ln(\frac{L^g}{S_T})} [e^{-g}(K_h - S_T)^+ - e^g(K_\ell - S_T)^+] \right\}. \quad (3.34)$$

For future use, we note from (3.4) that this payoff vanishes at $S_T = 0$ if $r \leq q$ and explodes otherwise.

When $r = q$. (3.34) implies that the sale of a one touch is hedged by simply buying $\frac{1 + e^{-g}}{K_h - K_\ell}$ puts struck at $K_h = L^g e^g$ and selling $\frac{1 + e^{-g}}{K_h - K_\ell}$ puts struck at $K_\ell = L^g e^{-g}$. The terminal payoff from this normalized and ratioed spread is:

$$\frac{1}{K_h - K_\ell} \left\{ (1 + e^{-g})(K_h - S_T)^+ - (1 + e^g)(K_\ell - S_T)^+ \right\}. \quad (3.35)$$

The payoff in (3.35) is one at $S_T = 0$, increases linearly with slope $\frac{e^g - e^{-g}}{K_h - K_\ell}$ for $S_T \in (0, K_\ell)$, decreases linearly with slope $-\frac{1 + e^{-g}}{K_h - K_\ell}$ for $S_T \in (K_\ell, K_h)$, and vanishes for $S_T \geq K_h \equiv L^g e^g$.

When $r = q$, or if the underlying is a forward price, then the weights in the hedge are independent of $\lambda_1$ and $\lambda_2$. It follows that the same semi-static hedge succeeds if $\lambda_2$ is generalized to be an arbitrary deterministic function of time, $\lambda_2(t)$, provided that $\lambda_1(t)$ is proportional to this function:

$$\lambda_1(t) = e^{-g} \lambda_2(t). \quad (3.36)$$

Similarly, the same semi-static hedge also succeeds if $\lambda_2$ is further generalized into a stochastic processes $\lambda_2$, which evolves independently of the two driving Poisson.
processes, provided that the process $\lambda_1 t$ is proportional:

$$\lambda_1 t = e^{-g} \lambda_2 t.$$  \hfill (3.37)

4. Jump to Default Case

4.1. Statistical and risk-neutral spot price process

In this section, we again assume that the barrier option is written on the spot price of some underlying asset. Under the statistical probability measure $\mathbb{P}$, we let $N_1$, $N_2$, and $N_3$ denote three independent standard Poisson processes, with positive jump arrival rates $\alpha_1$, $\alpha_2$, and $\alpha_3$ respectively. For given positive constants $g$ and $S_0$, we assume that the stochastic process governing the spot price of the underlying asset is given by:

$$S_t = S_0 e^{g(N_1 t - N_2 t)} 1(N_3 t = 0), \quad t \in [0, T].$$

These dynamics differ from those of the last section by allowing the spot price to jump to zero at an independent and exponentially distributed time. We refer to this event as a default since the spot price remains at zero afterwards. To compensate for carrying costs, risk, and the possibility of default, the three jump arrival rates $\alpha_1$, $\alpha_2$, and $\alpha_3$ must produce an expected exponential growth rate of $r - q + \pi + \alpha_3$ per unit time:

$$\alpha_1 (e^g - 1) + \alpha_2 (e^{-g} - 1) = r - q + \pi + \alpha_3.$$  \hfill (4.1)

Let $\lambda_1$, $\lambda_2$, and $\lambda_3$ denote risk-neutral arrival rates of the three Poisson processes. All three risk-neutral arrival rates must be positive and they are related by:

$$\lambda_1 (e^g - 1) + \lambda_2 (e^{-g} - 1) = r + \lambda_3 - q.$$  \hfill (4.2)

Solving (4.2) for $\lambda_2$ implies:

$$\lambda_2 = \frac{\lambda_1 e^g - r + \lambda_3 - q}{1 - e^{-g}}.$$  \hfill (4.3)

For future use, we note that:

$$\lambda_2 \geq \lambda_1 e^g \iff r + \lambda_3 - q \leq 0.$$  \hfill (4.4)

To infer the values of the three arrival rates, we assume that the initial rates on both a credit default swap (CDS) and a variance swap are known. We assume that the recovery rate on the bond insured by the CDS is known and hence knowledge of the CDS rate determines $\lambda_3$. For the variance swap, we assume that the possibility of default changes the terms from that of the last section. In particular, we now assume that the realized variance is calculated by squaring percentage returns rather than log price relatives. Thus, the payoff on a variance swap with fixed rate $V_0$ is:

$$\frac{1}{T} \int_0^T \left( \frac{dS_t}{S_t} \right)^2 - V_0^2.$$  \hfill (4.5)
In the current context:

\[ V_0^2 = \lambda_1(e^g - 1)^2 + \lambda_2(e^{-g} - 1)^2 + \lambda_3. \]  

(4.6)

Since \( \lambda_3 \) is known from the initial CDS rate, (4.2) and (4.6) constitute two linear equations in the two unknowns \( \lambda_1 \) and \( \lambda_2 \), which can be easily solved. The result is:

\[ \lambda_1 = \frac{V_0^2 - e^{-g}\lambda_3 - (e^{-g} - 1)(r - q)}{(e^g - 1)(e^g - e^{-g})} \]

\[ \lambda_2 = \frac{V_0^2 - e^g\lambda_3 - (e^g - 1)(r - q)}{(1 - e^{-g})(e^g - e^{-g})}. \]

To derive GPCS in the current context, we may use the law of iterated expectations:

\[ C(S_t e^{\ell}) = e^{-r(T-t)} E_t^Q \left( S_T - S_t e^{\ell} \right)^+ \]

\[ = e^{-(r+\lambda_3)(T-t)} E_t^Q \left( [S_T - S_t e^{\ell}]^+ | N_{3T} = 0 \right), \]

(4.7)

since the call payoff vanishes if \( N_{3T} > 0 \). As we are conditioning on no default to \( T \), we may repeat all of the steps used between (3.18) and (3.21) with the understanding that \( \lambda_1 \) and \( \lambda_2 \) are governed by (4.2) rather than (3.3). As a result, we have:

\[ C_t(S_t e^{\ell}) = e^{\ell} e^{-(r+\lambda_3)(T-t)} E_t^Q \left( \frac{S_T}{S_t} \right)^{\frac{1}{2} \ln \left( \frac{S_T}{S_t} \right)} (S_t e^{\ell} - S_T)^+ | N_{3T} = 0 \right). \]

(4.8)

To make progress, we now assume in this section that the right-hand side of (4.2) is nonpositive i.e.:

\[ r + \lambda_3 - q \leq 0. \]

(4.9)

In words, the carrying cost of the underlying asset (inclusive of default protection) is not positive. Put another way, we require that \( q \geq r \) and that the risk-neutral arrival rate of default should not be more than the difference:

\[ \lambda_3 \in [0, q - r]. \]

(4.10)

The motivation for this assumption is the equivalence (4.4), which further implies that:

\[ \ln \left( \frac{\lambda_2}{\lambda_1 e^{\ell}} \right) \geq 0. \]

(4.11)

As a consequence, the payoff \( \left( \frac{S_T}{S_t} \right)^{\frac{1}{2} \ln \left( \frac{S_T}{S_t} \right)} (S_t e^{-\ell} - S_T)^+ \) in (4.8) vanishes at \( S_T = 0 \) or equivalently when \( N_{3T} > 0 \). Hence under assumption (4.9), the conditional expectation in (4.8) simplifies into the following unconditional expectation:

\[ C_t(S_t e^{\ell}) = e^{\ell} e^{-(r+\lambda_3)(T-t)} E_t^Q \left( \frac{S_T}{S_t} \right)^{\frac{1}{2} \ln \left( \frac{S_T}{S_t} \right)} (S_t e^{\ell} - S_T)^+. \]

(4.12)

We refer to this result as Put Call Symmetry under Jump To Default (PCSJTD).
To apply PCSJTD to the problem of hedging the sale of a one touch, let $\tau_g \equiv \inf\{t \geq 0 : S_t = L_g\}$ denote the first time that $S$ jumps to the level $L_g$. We adopt the usual convention that the infimum of an empty set is infinite. Evaluating (4.12) at $t = \tau_g$ implies:

$$C_{\tau_g}(L^g e^\ell) = e^\ell e^{-r(T - \tau_g)} E_{\tau_g}^Q \left( S_T \right)^{\frac{1}{2} \ln(\frac{L^g e^\ell}{L^g e^\ell - \ell})} \left( S_T - L_T^g \right) + .$$

(4.13)

Let $f(S)$ denote the path-independent payoff (3.34) used to hedge the sale of a one touch in the last section:

$$f(S) \equiv \frac{1}{K_h - K_\ell} \left\{ (K_h - S)^+ - (K_\ell - S_T)^+ \right\} + \left( \frac{S_T}{L^g} \right)^{\frac{1}{2} \ln(\frac{L^g e^\ell}{L^g e^\ell - \ell})} \left[ e^{-g}(K_h - S)^+ - e^g(K_\ell - S)^+ \right],$$

(4.14)

where from (3.28) and (3.29), $K_\ell \equiv L^g e^{-g}$ is the lower strike in the normalized put spread and $K_h \equiv L^g e^g$ is the higher strike.

We now show that this payoff can also be used to hedge the sale of a one touch in the current context, so long as the nonpositive carry assumption (4.9) is holding. First, suppose that the barrier $L$ is never crossed by time $T$. Since $f$ vanishes for $S > L^g e^g$, the payoff evaluates to the desired value of zero. Second, suppose that the lower barrier $L$ is crossed by time $T$ and that the cross occurs by a jump to default. Under the nonpositive carry assumption (4.9), (4.11) implies that $f$ takes the value one at $S_T = 0$:

$$f(0) = 1.$$ 

(4.15)

As a result, if the barrier $L$ is crossed by a jump to default, then since $S$ stays at zero afterwards, the option portfolio used to create the payoff $f(\cdot)$ produces the desired payoff of one at $T$. Finally, suppose instead that the lower barrier $L$ is first crossed by a jump to the level $L^g$. Then (4.13) implies that the options used to create the last term in (4.14) can be liquidated and the proceeds can be used to buy a normalized call spread as before. We conclude from no arbitrage that the one touch is initially valued at the cost of creating the payoff $f(\cdot)$ in (4.14).

5. Summary and Extensions

We showed that the payoff to barrier options can be replicated when the underlying price process is driven by the difference of two independent Poisson processes. The replicating strategy employed simple semi-static positions in co-terminal standard options. We considered three different dynamical settings, increasing in complexity and financial realism. In the first two settings, we showed that when the underlying of the barrier option has no carrying cost, the same semi-static trading strategy...
continues to replicate under more general dynamics, which require only partial specification. Finally, we showed that previous results obtained for continuous processes arise as limits of our analysis.

It would be interesting to attempt to extend these results to the case where there is uncertainty in the landing level at the barrier crossing time. A tractable candidate for the jump size distribution is the exponential distribution. If we condition on the barrier crossing time, then the memoryless property implies that the event that the underlying jumps from somewhere above the barrier to somewhere below it has the same probability as the event that the underlying is at the barrier and jumps below. In the interests of brevity, this extension and others are left to future research.

Appendix: Spanning Put Portfolio

Recall from (3.20) that in the geometric case, a call struck at $S_t e^\ell$ has the same value as a claim with the path-independent payoff:

$$e^{\ell} \left( \frac{S_T}{S_t} \right)^{2p-1} (S_t e^{-\ell} - S_T)^+, \quad (A.1)$$

where $\ell$ is any real constant and $p$ is given in (3.11). In this appendix, we find the static position in bonds and options that creates this payoff. We find that the payoff is spanned by a portfolio of puts struck at $S_t e^{-\ell}$ and below. The payoff in (A.1) is clearly path-independent, continuous, positive for $S_T < S_t e^{-\ell}$, and zero for $S_T > S_t e^{-\ell}$. In general, if a payoff function $f(Z)$ is $C^2$, then one can Taylor expand about some point $\kappa$. As shown in Carr and Madan [1], one can employ a form of Taylor series with remainder in which the remainder term is conveniently expressed in terms of the payoffs from puts and calls. For our application, the payoff function is $C^2$ on its domain $Z > 0$, except that it is not differentiable at $S_t e^{-\ell}$.

If a function $f(Z)$ is $C^2$ on its domain $\mathbb{R}$, except that it is not differentiable at a single point $\kappa$, then it is easily shown that the first order Taylor series with second order remainder is:

$$f(Z) = f(\kappa) + \lim_{Z \downarrow \kappa} f'(Z)(Z - \kappa)^+ + \lim_{Z \uparrow \kappa} f'(Z)(\kappa - Z)^+$$

$$+ \int_{-\infty}^{\kappa} f''(K)(K - Z)^+ dK + \int_{\kappa}^{\infty} f''(K)(Z - \kappa)^+ dK. \quad (A.2)$$

The right-hand side of (A.2) is the payoff arising from a static position consisting of:

- $f(\kappa)$ bonds, with each bond paying $1$ at $T$.
- $\lim_{Z \downarrow \kappa} f'(Z)$ calls struck at $\kappa$.
- $-\lim_{Z \uparrow \kappa} f'(Z)$ puts struck at $\kappa$.
- $f''(K)dK$ puts struck at $K$ for each $K \in (-\infty, \kappa)$.
- $f''(K)dK$ calls struck at $K$ for each $K \in (\kappa, \infty)$. 
To apply this result to our setting, we choose the function $f(Z)$ as:

$$f(Z) = e^\ell \left( \frac{Z}{S_t} \right)^{2p-1} (S_te^{-\ell} - Z)^+$$  \hspace{1cm} (A.3)

with domain $Z > 0$ and fixed parameters $S_t$, $p$, and $\ell$. If we also choose $\kappa = S_te^{-\ell}$, then (A.2) simplifies into:

$$f(Z) = -\lim_{Z \uparrow S_t e^{-\ell}} f'(Z)(S_t e^{-\ell} - Z)^+ + \int_0^{S_t e^{-\ell}} f''(K)(K - Z)^+ dK.$$  \hspace{1cm} (A.4)

Hence, the position in bonds has disappeared since $f(S_t e^{-\ell}) = 0$ from continuity. The positions in calls have also disappeared since $\lim_{Z \downarrow S_t e^{-\ell}} f'(Z) = 0$ and $f''(Z) = 0$ for $Z > S_t e^{-\ell}$.

Although $f$ is not differentiable at $Z = S_t e^{-\ell}$, we may formally differentiate (A.3) w.r.t. $Z$ and treat the result as a generalized function. Performing the indicated differentiation implies:

$$f'(Z) = e^\ell 2p - 1 \left( \frac{Z}{S_t} \right)^{2p-2} (S_t e^{-\ell} - Z)^+ - e^\ell \left( \frac{Z}{S_t} \right)^{2p-1} \mathcal{H}(S_t e^{-\ell} - Z),$$  \hspace{1cm} (A.5)

where $\mathcal{H}(\cdot)$ denotes the Heaviside function. Evaluating (A.5) as $Z \uparrow S_t e^{-\ell}$ implies:

$$\lim_{Z \uparrow S_t e^{-\ell}} f'(Z) = -e^\ell e^\ell(2p-1) = -e^{2\ell(1-p)}.$$  \hspace{1cm} (A.6)

Since $(S_t e^{-\ell} - Z)^+ = \mathcal{H}(S_t e^{-\ell} - Z)(S_t e^{-\ell} - Z)$, the first derivative in (A.5) can be written in the following simpler form:

$$f'(Z) = \left( \frac{Z}{S_t} \right)^{2p-2} \left( 2p - 1 + \frac{Z}{S_t} 2pe^\ell \right),$$  \hspace{1cm} (A.7)

for $Z < S_t e^{-\ell}$, and $f'(Z) = 0$ for $Z > S_t e^{-\ell}$. Differentiating (A.7) w.r.t. $Z$ implies:

$$f''(Z) = 2p - 2 \left( \frac{Z}{S_t} \right)^{2p-3} \left( 2p - 1 + \frac{Z}{S_t} 2pe^\ell \right) + \left( \frac{Z}{S_t} \right)^{2p-2} \frac{2pe^\ell}{S_t},$$  \hspace{1cm} (A.8)

for $Z < S_t e^{-\ell}$, and $f''(Z) = 0$ for $Z > S_t e^{-\ell}$. Simplifying this expression:

$$f''(Z) = \left( \frac{Z}{S_t} \right)^{2p-2} \left( \frac{(2p-1)(2p-2)}{Z} + \frac{(2p-1)2pe^\ell}{S_t} \right)$$

$$= (2p - 1)2 \left( \frac{Z}{S_t} \right)^{2p-2} \left[ \frac{p - 1}{Z} + \frac{pe^\ell}{S_t} \right],$$  \hspace{1cm} (A.9)

for $Z < S_t e^{-\ell}$, and $f''(Z) = 0$ for $Z > S_t e^{-\ell}$. Substituting (A.6) and (A.9) in (A.4) leads to:

$$f(Z) = e^{2\ell(1-p)}(S_t e^{-\ell} - Z)^+ + (2p - 1)2$$

$$\times \int_0^{S_t e^{-\ell}} \left\{ \left( \frac{K}{S_t} \right)^{2p-2} \left[ \frac{p - 1}{K} + \frac{pe^\ell}{S_t} \right] \right\}(K - Z)^+ dK.$$  \hspace{1cm} (A.10)
Thus, the desired payoff is spanned by holding:

- $e^{2\ell(1-p)}$ puts struck at $S_t e^{-\ell}$, and
- $(2p - 1)2\{(\frac{S_t}{K})^{2p-2}[\frac{e^{\ell}}{K} + \frac{e^{2\ell}}{S_t}]\}dK$ puts for all strikes $K \in (0, S_t e^{-\ell})$.

We note that the last term in (A.10) vanishes if $r = q$, since $2p - 1$ evaluates to zero in that case. Thus, when $r = q$, the payoff simplifies into a multiple of the payoff produced by a single put struck at $S_t e^{-\ell}$:

$$f(Z) = e^{\ell}(S_t e^{-\ell} - Z)^+.$$  \hspace{1cm} (A.11)

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References