Stochastic Lagrangian Modeling of Traffic Dynamics

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Abstract

This paper proposes a new stochastic model of traffic dynamics in Lagrangian coordinates. The source of uncertainty in the proposed model is parametric. Consequently, uncertainty in the model can be interpreted as capturing heterogeneity in the driver population. It also results in smooth trajectories in a stochastic context, which is in agreement with real-world traffic dynamics and, thereby, overcoming issues with aggressive oscillation typically observed in sample paths of stochastic traffic flow models. A stochastic version of Newell’s car-following model is utilized. The mean dynamics of the model are derived as the limiting dynamics of an ensemble averaged process as the ensemble size goes to infinity. Covariance dynamics are also derived using a Gaussian approximation of the stochastic system. Numerical examples are provided to illustrate convergence of ensemble averaged process to the mean dynamics, as well as to illustrate the behavior of covariance dynamics. A data assimilation example is also given. Results show that the model can be used to estimate aggregated measures of traffic conditions with reasonable accuracy.

Keywords:
Lagrangian coordinates, heterogeneous drivers, car following, mean dynamics, variability, hydrodynamic limits, uncertainty quantification

1. Introduction

Efficient traffic operation and optimization require knowledge of prevailing traffic conditions. The Lighthill, Whitham and Richards traffic flow...
model (Lighthill and Whitham, 1955; Richards, 1956) (the LWR model) has been widely applied in estimation and prediction of traffic states on both freeways and high-speed intersections. The model is formulated using traditional spatial-temporal (Eulerian) coordinates and is suitable for state estimation with point sensor measurements (macroscopic data, e.g., traffic volume, speeds). Data from probe vehicles or connected vehicles (microscopic data, e.g., vehicle trajectories) are becoming increasingly available. Traffic flow models that are able to effectively utilize such data are of greater interest in modern applications. A simple way of interfacing between the microscopic and the macroscopic worlds is via coordinate transformations. Indeed, this was done by Daganzo (Daganzo, 2005a,b) and later extended by Leclercq et al. (2007). The former proposes a variational formulation of the LWR model in Eulerian coordinates while the later proposes to formulate the model in Lagrangian coordinates. More recently, Hamilton-Jacobi based formulations of traffic flow have appeared in the literature (Claudel and Bayen, 2010; Friesz et al., 2013) and Laval and Leclercq (2013) applied the theory to formulate first-order models in three different coordinate systems, namely the traditional Eulerian coordinates and two variants of the Lagrangian coordinates. Proposed solutions schemes for the deterministic Lagrangian models include both variational techniques and the Godunov scheme using a triangular fundamental diagram. Specifically, the Godunov scheme in Lagrangian coordinates simplifies to an upwind scheme, enabling more efficient application of data assimilation methods (Duret and Yuan, 2017; Laval and Leclercq, 2013; Yuan et al., 2012).

Though deterministic traffic flow models and their solution methods have been extensively studied in the literature, stochastic models of traffic flow are still in an immature stage of development, particularly with regards to issues related to the physical relevance of the stochastic dynamics. The main culprit is the dominance of time-stochasticity (or noise) in stochastic models, mostly developed in Eulerian coordinates (Gazis and Knapp, 1971; Szeto and Gazis, 1972; Muñoz et al., 2003; Gazis and Liu, 2003; Wang and Papageorgiou, 2005; Boel and Mihaylova, 2006; Wang et al., 2007; Work et al., 2008; Di et al., 2010; Sumalee et al., 2011; Blandin et al., 2012; Jabari and Liu, 2012, 2013), but also in Lagrangian coordinates (Yuan et al., 2012, 2015; Chu et al., 2016). This results in sample paths prone to aggressive oscillation in the time dimension. The interpretation of these oscillations is (unreasonably) aggressive acceleration and deceleration.

This paper addresses the physical relevance issue of stochastic traffic dy-
dynamics via a new stochastic Lagrangian model of traffic flow. The source of uncertainty in the model is parametric in the same sense presented in (Jabari et al., 2014). The interpretation of this form of uncertainty is heterogeneity in the driving population. We utilize a stochastic version of Newell’s car-following model\(^1\) (Newell, 2002). The model is discrete in the vehicle index dimension; changing the vehicle index scaling, the model can be considered as a (first-order) Godunov scheme for solving continuum Lagrangian traffic models.

We demonstrate that the resulting sample paths do not contain the oscillatory behavior above. To ensure computational tractability, we derive (deterministic) mean and covariance dynamics, which can be used in traffic estimation (and control) applications.

This paper is organized as follows: Sec. 2 presents Newell’s simplified car-following model with heterogeneous drivers along with the stochastic model of the car-following dynamics. In Sec. 3, we derive the mean and covariance dynamics of the stochastic system. Sec. 4 presents numerical example and Sec. 5 concludes the paper.

2. The traffic dynamics

2.1. Heterogeneous model

We assume a system with \( \mathbb{N}_+ \ni N < \infty \) vehicles numbered in descending order of position; that is vehicle \( n = 0 \) is the leader of all vehicles, \( n = 1 \) is the immediate follower, and so on until \( n = N \), which is the last vehicle in the system. We assume a finite time horizon \( \mathbb{R}_+ \ni T < \infty \). Let \( x_n(t) \) and \( v_n(t) \) denote the position and speed of vehicle \( n \) at time \( t \in [0,T] \), respectively. We denote the spacing between vehicle \( n \) and their leader, \( n-1 \) by \( s_n(t) \equiv x_{n-1}(t) - x_n(t) \). Since

\[
x_n(t + \Delta t) = x_n(t) + \int_t^{t+\Delta t} v_n(\tau) \, d\tau,
\]

we have that

\[
s_n(t + \Delta t) = s_n(t) + \int_t^{t+\Delta t} (v_{n-1}(\tau) - v_n(\tau)) \, d\tau.
\]

\(^1\)It is notable that Newell’s model was intended to be stochastic, but is often treated as a deterministic theory.
We adopt Newell’s speed-spacing relation (Newell, 2002) with heterogeneous parameters; see Fig. 1 for an example. The speed-spacing relation is
given for vehicle \( n \) by

\[
V_n(s) = \min \left\{ v_{n,f}, w_n(k_n j s - 1)^+ \right\},
\]

(3)

where \((\cdot)^+ \equiv \max\{\cdot, 0\}\), \( v_{n,f} > 0 \) is the free-flow speed of vehicle \( n \) (in km/hr), \( k_n j > 0 \) is the jam density of vehicle \( n \) (in veh/km), and \( w > 0 \) is the (absolute) speed of the backward wave of vehicle \( n \) (in km/hr). Then,

\[
s_n(t + \Delta t) = s_n(t) + \Delta t \left( V_{n-1}(s_{n-1}(t)) - V_n(s_n(t)) \right),
\]

(4)

where, in a setting with homogeneous drivers, i.e., \( v_{n,f} = v_f, w_n = w, k_{n,j} = k_j \) for all \( n \), \( \Delta t \) is chosen as the reaction time (or speed adaptation time) \( \tau_r = (wk_j)^{-1} \). In a setting with heterogeneous drivers, we set \( \Delta t = \min_n \tau_{n,r} = \min_n(w_n k_{n,j})^{-1} \) to avoid violations of the Courant-Friedrichs-Lewy (CFL) condition, while also mitigating numerical diffusion.

2.2. Parametric uncertainty and stochastic dynamics

To introduce stochasticity, we let the parameters be random variables. We interpret this as uncertainty about the driver characteristics. To differentiate the stochastic case from the deterministic case, we write the (stochastic) parameters as functions of \( \omega \), where \( \Omega \ni \omega \) is the random space. We assume the
random triples (the parameters), \(\{v_{n,f}(\omega), w_{n}(\omega), k_{n,j}(\omega)\}_{n=1}^{N}\) are independent and identically distributed. The speed-spacing relation pertaining to vehicle \(n\) is written as:

\[
V_n(s, \omega) = \min\left\{v_{n,f}(\omega), w_{n}(\omega)\left(k_{n,j}(\omega)s - 1\right)^+\right\}.
\]

(5)

The stochastic dynamical model evolves according to

\[
s_n(t + \Delta t, \omega) = s_n(t, \omega) + \Delta t \left(V_{n-1}\left(s_{n-1}(t, \omega), \omega\right) - V_{n}\left(s_n(t, \omega), \omega\right)\right).
\]

(6)

We denote the distribution functions of free-flow speed, backward wave speed, and jam density, respectively, by \(F_f(\cdot)\), \(F_w(\cdot)\), and \(F_j(\cdot)\). To ensure that the speed-spacing relations are physically reasonable, the supports of the three distributions must be bounded from both above and below. That is, we assume the existence of constants, \(0 < v_{f}^{\text{min}} < v_{f}^{\text{max}} < \infty\), \(0 < w^{\text{min}} < w^{\text{max}} < \infty\), and \(0 < k_{j}^{\text{min}} < k_{j}^{\text{max}} < \infty\), such that \(\mathbb{P}(v_{f}(\omega) \in [v_{f}^{\text{min}}, v_{f}^{\text{max}}]) = 1\), \(\mathbb{P}(w_{1}(\omega) \in [w^{\text{min}}, w^{\text{max}}]) = 1\), and \(\mathbb{P}(k_{j}(\omega) \in [k_{j}^{\text{min}}, k_{j}^{\text{max}}]) = 1\). This is also needed to ensure a finite minimum reaction time in the stochastic context. Essentially, we set \(\Delta t = (w^{\text{max}}k_{j}^{\text{max}})^{-1}\). Algorithm 1 illustrates how to simulate a single sample path of the process. Essentially, the algorithm randomly generate random realization of the parameters, one realization per vehicle and then simulates a heterogeneous driving environment.

3. Mean dynamics and variability

A consequence of the non-linearity in the stochastic model (6), via the non-linearity in \(V_n(\cdot, \omega)\), applications such as data assimilation and traffic control will require some form of sampling. In this section, we derive the mean dynamics that would arise from an infinitely sized sample along with the dynamics of variance of the stochastic process. To this end, let \(M\) denote an ensemble size and \(m = 1, \cdots, M\) the index of the sample path in the ensemble. We denote the \(m\)th spacing process by \(s^m_n(\cdot, \omega)\) for \(n = 1, \cdots, N\). Below, we derive the ensemble-averaged and the deviation processes.

3.1. Mean dynamics

Consider the ensemble-averaged processes \(\left\{s^M_n(\cdot, \omega)\right\}_{n=1}^{N}\) which evolves as

\[
s^M_n(t, \omega) = s_n(0) + \frac{1}{M} \sum_{m=1}^{M} \int_0^t \left(V_{n-1}\left(s^M_{n-1}(\tau, \omega), \omega\right) - V_{n}\left(s^M_n(\tau, \omega), \omega\right)\right) d\tau
\]

(7)
Algorithm 1: Simulating a single sample path of the process

Initialize: $N, T, \Delta t, x_0(\cdot), v_0(\cdot), \{s_n(0)\}_{n=1}^{N}, F_t(\cdot), F_w(\cdot), F_j(\cdot)$

Iterate:
1: for $n = 1$ to $n \leq N$ do
2: $U_1, U_2, U_3 \text{ i.i.d. Uniform}[0,1]$
3: $v_{n,1} := F_{v_1}^{-1}(U_1)$
4: $w_n := F_w^{-1}(U_2)$
5: $k_{n,j} := F_j^{-1}(U_3)$
6: end for
7: for $j := 0$ to $j < \lfloor \frac{T}{\Delta t} \rfloor$ do
8: for $n := 1$ to $n < N$ do
9: /* $\{V_n(\cdot)\}$ are calculated using (3) */
10: $s_n((j+1)\Delta t) := s_n(j\Delta t) + \Delta t(V_{n-1}(s_{n-1}(j\Delta t)) - V_n(s_n(j\Delta t)))$
11: $x_n(j\Delta t) := x_{n-1}(j\Delta t) - s_n(j\Delta t)$
12: end for
13: end for
14: for $n = 1, \cdots, N$. Without loss of generality, we have assumed that the initial spacings are deterministic. For any $s \geq 0$, the speed-spacing relations are given by

$$V^m_n(s, \omega) = \min\{v^m_n(\omega), w^m_n(\omega)(k^m_n(\omega)s - 1)^+, \}$$

where the triples $(v^m_n(\omega), w^m_n(\omega), k^m_n(\omega))$ are i.i.d. (in both $n$ and $m$) with distribution functions $F_t(\cdot), F_w(\cdot)$ and, $F_j(\cdot)$. From the strong law of large numbers we have that

$$\frac{1}{M} \sum_{i=1}^{M} V^m_n(s, \omega) \xrightarrow{M \to \infty} \mathbb{V}(s) \text{ almost surely},$$

where $\mathbb{V}(s) \equiv \mathbb{E}V^1_1(s, \omega)$. Note that $\mathbb{V}(s) \neq \min\{v_f, \bar{w}(k_j s - 1)^+\}$ (for some appropriately chosen constants $v_f$, $\bar{w}$, and $k_j$). The right-hand side is a percentile speed-spacing relation (typically, a 0.5-percentile or equilibrium relation), while $\mathbb{V}(s)$ is a mean speed-spacing relation; see (Jabari et al., 2014) for more details. An example comparison is shown in Fig. 2. We will not attempt to derive an expression for $\mathbb{V}(\cdot)$. Instead, we will use an empirical approximation: Let $\{(v_{m,f}, w_m, k_{m,j}), m = 1, \cdots, M\}$ be an i.i.d.
Fig. 2: Mean relation, $\bar{V}(\cdot)$ vs. percentile relation.

random sample of the parameters. Then, for any $s$ and a sufficiently large $M$, $\bar{V}(s)$ is very well approximated by

$$
\bar{V}(s) \approx \frac{1}{M} \sum_{m=1}^{M} \min \left\{ v_{m,f}, w_{m}(k_{m,i} s - 1)^+ \right\}.
$$

Note that the pseudo-random parameter triples need only be determined once, so that the approximation (10) can be established off-line (as part of a preprocessing step). Computational efficiency can be further improved by means of sparse approximations (see (Dilip et al., 2017; Jabari et al., 2017) for example).

To establish convergence we utilize the uniform norm, which for the sake of completeness we review next: For a process (with continuous sample paths) $z(\cdot) : [0, T] \to \mathbb{R}^M$ with components $\{z_i(\cdot)\}_{i=1}^{M}$, i.e., $z(\cdot) = [z_1(\cdot) \cdots z_M(\cdot)]^T$, the uniform norm is defined as

$$
\|z(\cdot)\|_T \equiv \sup_{0 \leq t \leq T} \max_{1 \leq i \leq M} |z_i(t)|.
$$

Whenever

$$
\|z^{(\nu)}(\cdot) - z(\cdot)\|_T \xrightarrow{\nu \to \infty} 0 \text{ for all } t \in [0, T]
$$

holds for a sequence of processes $z^{(1)}(\cdot), z^{(2)}(\cdot), \cdots$ and a limit process $z(\cdot)$, the sequence is said to converge uniformly on compact sets (u.o.c.), a form of strong convergence (almost sure convergence on $[0, T]$).
To establish the mean dynamics, we first write (7) in vector form:

\[ s^M(t, \omega) = s(0) + \frac{1}{M} \sum_{m=1}^{M} \int_0^t D V^m(s^m(\tau, \omega), \omega) d\tau, \]

(13)

where \( s^M(\cdot, \omega) \equiv [s^M_1(\cdot, \omega) \cdots s^M_N(\cdot, \omega)]^\top \), \( V^m(s, \omega) \equiv [V^m_1(s_1, \omega) \cdots V^m_N(s_N, \omega)]^\top \), and \( D : \mathbb{R}^N \to \mathbb{R}^N \) is a linear operator defined by \( D V^m(s(\cdot), \omega) \equiv [v_0(\cdot) - V^m_1(s_1(\cdot), \omega) \cdots V^m_N(s_N(\cdot), \omega)]^\top \). Next, consider the deterministic (mean) processes given, for each \( n \in \{1, \cdots, N\} \), by

\[ \hat{s}_n(t) = s_n(0) + \int_0^t \left( \hat{V}(\hat{s}_{n-1}(\tau)) - \hat{V}(\hat{s}_n(\tau)) \right) d\tau. \]

(14)

Define \( \hat{s}(\cdot) \equiv [\hat{s}_1(\cdot) \cdots \hat{s}_N(\cdot)]^\top \) and \( \hat{V}(\hat{s}(\cdot)) \equiv [\hat{V}(\hat{s}_1(\cdot)) \cdots \hat{V}(\hat{s}_N(\cdot))]^\top \).

Then

\[ \hat{s}(t) = s(0) + \int_0^t D \hat{V}(\hat{s}(\tau)) d\tau. \]

(15)

It can be easily demonstrated that \( V^m(s, \omega) \) is Lipschitz continuous (in \( s \)); let \( 0 \leq K < \infty \) denote the Lipschitz constant: \( K \) is the smallest constant such that for all \( m = 1, \cdots, M \) and any \( s_1, s_2 \geq 0 \)

\[ \| V^m(s_1, \omega) - V^m(s_2, \omega) \| \leq K \| s_1 - s_2 \|. \]

(16)

Since the supports of the parameters are bounded from above and below, \( K \) exists and is easy to determine. We now prove that the following holds:

\[ \| s^M(\cdot, \omega) - \hat{s}(\cdot) \|_t \rightarrow 0 \text{ for all } t \in [0, T]. \]

(17)

First, it follows immediately from (9) that

\[ \left\| \frac{1}{M} \sum_{m=1}^{M} V^m(z(\cdot), \omega) - \hat{V}(z(\cdot)) \right\|_t \rightarrow 0 \text{ for all } t \in [0, T] \]

(18)

for any (vector) process \( z(\cdot) \) with continuous sample paths.

Next, using the triangle inequality and noting that, for any \( t \), \( \hat{V}(\hat{s}(t)) = M^{-1} \sum_{m=1}^{M} \hat{V}(\hat{s}(t)) \), we have that

\[ \| s^M(\cdot, \omega) - \hat{s}(\cdot) \|_t \leq \left\| \int_0^t \frac{1}{M} \sum_{m=1}^{M} D \left( V^m(s^m(\tau, \omega), \omega) - \hat{V}(\hat{s}(\tau)) \right) d\tau \right\|_t. \]

(19)
Adding and subtracting $M^{-1} \sum_{m=1}^{M} D V^m(\hat{s}(\tau), \omega)$ to the right-hand side (inside the integral), we have that

$$\| s^m(\cdot, \omega) - \hat{s}(\cdot) \|_t \leq \left\| \int_0^t \frac{1}{M} \sum_{m=1}^{M} D \left( V^m(s^m(\tau, \omega), \omega) - V^m(\hat{s}(\tau), \omega) \right) d\tau \right\|_t + \left\| \int_0^t \frac{1}{M} \sum_{m=1}^{M} D \left( V^m(\hat{s}(\tau), \omega) - \bar{V}(\hat{s}(\tau)) \right) d\tau \right\|_t. \quad(20)$$

That the second term on the right-hand side converges to zero follows from (18) and the linearity of $D$ (in accord with the continuous mapping theorem).

To simplify notation, let $\epsilon^M(t)$ denote the second term. We have that

$$\epsilon^M(t) = \left\| \int_0^t \frac{1}{M} \sum_{m=1}^{M} D \left( V^m(s^m(\tau, \omega), \omega) - V^m(\hat{s}(\tau), \omega) \right) d\tau \right\|_t \to 0 \quad(21)$$

for all $t \in [0, T]$. Applying the triangle inequality and since $D$ is linear, the first term on the right-hand side of (20) is bounded from above by

$$\int_0^t \left\| \frac{1}{M} \sum_{m=1}^{M} D \left( V^m(s^m(\tau, \omega), \omega) - V^m(\hat{s}(\tau), \omega) \right) \right\|_\tau d\tau. \quad(22)$$

Let $\hat{K} < \infty$ denote the Lipschitz constant of $D V^m(\cdot, \omega)$, then (22) is bounded from above by

$$\int_0^t \left\| \frac{\hat{K}}{M} \sum_{m=1}^{M} (s^m(\cdot, \omega) - \hat{s}(\cdot)) \right\|_\tau d\tau = \hat{K} \int_0^t \| s^M(\cdot, \omega) - \hat{s}(\cdot) \|_\tau d\tau. \quad(23)$$

Hence,

$$\| s^M(\cdot, \omega) - \hat{s}(\cdot) \|_t \leq \epsilon^M(t) + \hat{K} \int_0^t \| s^M(\cdot, \omega) - \hat{s}(\cdot) \|_\tau d\tau. \quad(24)$$

Applying the Bellman-Grönwall inequality, we have that

$$\| s^M(\cdot, \omega) - \hat{s}(\cdot) \|_t \leq \epsilon^M(t) e^{\hat{K}t}. \quad(25)$$

For all $t \in [0, T]$, (17) follows from (21) as $M \to \infty$. This completes the proof.
3.2. Hydrodynamic limit

The result above can be generalized to any vehicle size scaling. In essence, we have thus far assumed that $\Delta n = 1$. This can be easily generalized to any $\Delta n$: The ensemble averaged process becomes

$$s^M(t, \omega) = s(0) + \frac{1}{M\Delta n} \sum_{m=1}^{M} \int_0^t D V^m(s^m(\tau, \omega), \omega) d\tau.$$  \hspace{1cm} (26)

In this case, we have $\lfloor N/\Delta n \rfloor$ vehicles in the system ($\lfloor \cdot \rfloor$ is the floor function). The limit spacing process can be derived using the same procedure presented above. It is given, for $n \in \{1, \cdots, \lfloor N/\Delta n \rfloor \}$, by

$$\hat{s}_n(t) = s_n(0) + \frac{1}{\Delta n} \int_0^t \left( \nabla(\hat{s}_{n-1}(\tau)) - \nabla(\hat{s}_n(\tau)) \right) d\tau.$$ \hspace{1cm} (27)

In vector form ($\hat{s}(t) \in \mathbb{R}^{\lfloor N/\Delta n \rfloor}$):

$$\hat{s}(t) = s(0) + \frac{1}{\Delta n} \int_0^t D \nabla(\hat{s}(\tau)) d\tau.$$ \hspace{1cm} (28)

This is a deterministic process that converges as $\Delta n \to 0$ to a conservation law in Lagrangian coordinates:

$$\frac{\partial \hat{s}(n, t)}{\partial t} + \frac{\partial \nabla(\hat{s}(n, t))}{\partial n} = 0,$$ \hspace{1cm} (29)

where $\hat{s}(n, t)$ is a process in which $n$ is continuous and the speed relation $\nabla(\cdot)$ is a mean relation and not the traditional equilibrium relation used in the literature.

3.3. Second-order stochastic limit

Consider the (amplified) deviation process

$$\delta^M(t, \omega) \equiv \sqrt{M} \left( s^M(t, \omega) - \hat{s}(t) \right).$$ \hspace{1cm} (30)

The limit of $\delta^M(t, \omega)$ for each $t \in [0, T]$ is equivalent (in distribution) to the limit of the difference of the integrals in (26) and (28). We write this (prior
to taking the limit) as

\[ \int_0^t \frac{1}{\sqrt{M \Delta n}} \sum_{m=1}^M D \left( \mathbf{V}^m (s^m(\tau), \omega) - \mathbf{V}(\bar{s}(\tau)) \right) d\tau \]

\[ = \int_0^t \frac{1}{\sqrt{M \Delta n}} \sum_{m=1}^M D \left( \mathbf{V}^m (s^m(\tau), \omega) - \mathbf{V}(s^m(\tau), \omega)) \right) d\tau \]

\[ + \int_0^t \frac{1}{\sqrt{M \Delta n}} \sum_{m=1}^M D \left( \mathbf{V}^m (s^m(\tau), \omega) - \mathbf{V}(\bar{s}(\tau)) \right) d\tau, \tag{31} \]

where we added and subtracted \( \int_0^t \frac{1}{\sqrt{M \Delta n}} \sum_{i=1}^\nu D \mathbf{V}(s^M(\tau, \omega)) d\tau \). Applying the (generalized) continuous mapping theorem (Whitt, 2002), the first term on the right-hand side converges in distribution to

\[ \int_0^t \frac{1}{\Delta n} D \Sigma_{1/2}^{1/2} (\bar{s}(\tau)) \xi(\omega) d\tau, \tag{32} \]

where \( \Sigma_{1/2}^{1/2} (\bar{s}(\tau)) \) is a diagonal matrix with diagonal elements given by

\[ \Sigma_{n,n}^{1/2} (\bar{s}(t)) = \sqrt{\text{Var}(V_n^{1/2}(\bar{s}_n(t), \omega))}. \tag{33} \]

This can be calculated (off-line) using an empirical approximation as in (10) (using the same pseudo-random parameter sample). Let \( \tilde{\delta}(\cdot, \omega) \) denote the (weak) limit of \( \delta^M(\cdot, \omega) \) as \( M \to \infty \). Note that \( \tilde{\delta}(0, \omega) = 0 \) in accordance with our assumptions on the initial conditions. We have by definition, (30), that \( \hat{s}(\cdot) + \frac{1}{\sqrt{M}} \delta^M(\cdot, \omega) = s^M(\cdot, \omega) \). We can then write the second term on the right-hand side of (31) as

\[ \int_0^t \frac{1}{\sqrt{M \Delta n}} \sum_{m=1}^M D \left( \mathbf{V}(\bar{s}(\tau) + M^{-1/2} \delta^M(\tau, \omega)) - \mathbf{V}(\bar{s}(\tau)) \right) d\tau. \tag{34} \]

In accord with (17), we have that \( M^{-1/2} \delta^M(t, \omega) \to 0 \) almost surely as \( \nu \to \infty \). Consequently, (34) converges (weakly) to

\[ \frac{1}{\Delta n} \int_0^t G(\bar{s}(\tau)) \tilde{\delta}(\tau, \omega) d\tau, \tag{35} \]
where \( G(\hat{s}(t)) \) is an \( N \times N \) matrix valued function given by

\[
G(\hat{s}(t)) = \begin{bmatrix}
-\frac{dV(\hat{s}_1(t))}{ds} & 0 & 0 & \cdots & 0 & 0 \\
\frac{dV(\hat{s}_1(t))}{ds} & -\frac{dV(\hat{s}_2(t))}{ds} & 0 & \cdots & 0 & 0 \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & -\frac{dV(\hat{s}_{N-1}(t))}{ds} & -\frac{dV(\hat{s}_N(t))}{ds}
\end{bmatrix}.
\] (36)

The derivatives above are to be interpreted as weak derivatives in the following sense:

\[
\frac{dV(s)}{ds} = \frac{1}{M} \sum_{m=1}^{M} \frac{d}{ds} \min \{ v_{m,f}, w_m(k_{m,j}s - 1)^+ \},
\] (37)

where

\[
\frac{d}{ds} \min \{ v_{m,f}, w_m(k_{m,j}s - 1)^+ \} = \begin{cases}
0, & s \in (0, \frac{1}{k_{m,j}}) \cup (\frac{v_{m,f} + w_m}{w_m k_{m,j}}, \infty) \\
\frac{1}{2} v_{m,f}, & s = \frac{v_{m,f} + w_m}{w_m k_{m,j}} \\
\frac{1}{2} w_m k_{m,j}, & s = \frac{v_{m,f} + w_m}{w_m k_{m,j}} \\
w_m k_{m,j}, & \text{otherwise}
\end{cases}.
\] (38)

From (32) and (35), we have that

\[
\tilde{\delta}(t, \omega) = \tilde{\delta}(0) + \frac{1}{\Delta n} \int_0^t \left( G(\hat{s}(\tau)) \tilde{\delta}(\tau, \omega) + D \Sigma_{1/2}(\hat{s}(\tau)) \xi(\omega) \right) d\tau
\] (39)

or in differential form

\[
\frac{d\tilde{\delta}(t, \omega)}{dt} = \frac{1}{\Delta n} \left( G(\hat{s}(t)) \tilde{\delta}(t, \omega) + D \Sigma_{1/2}(\hat{s}(t)) \xi(\omega) \right).
\] (40)

The latter is a linear matrix (random) differential equation and has a closed form solution given by

\[
\tilde{\delta}(t, \omega) = \Phi(t) \left( \tilde{\delta}(0, \omega) + \frac{1}{\Delta n} \int_0^t \Phi^{-1}(\tau) D \Sigma_{1/2}(\hat{s}(\tau)) \xi(\omega) d\tau \right),
\] (41)
where $\Phi(\cdot)$ is the fundamental matrix, that is, it is the solution of

$$\frac{d\Phi(t)}{dt} = \frac{1}{\Delta n} G(\tilde{s}(t)) \Phi(t)$$

(42)

with initial condition $\Phi(0) = I$, where $I$ is the $N \times N$ identity matrix. The above implies that $\tilde{\delta}(\cdot, \omega)$ is a Gaussian process with (deterministic) covariance dynamics, $P(\cdot)$, given by $P(t) = E\tilde{\delta}(t, \omega)\tilde{\delta}(t, \omega)\top$. To avoid the need to compute the fundamental matrix, $\Phi(\cdot)$, the evolution of the covariance matrix can be calculated by taking the time derivative of $Q(\cdot) \equiv P\tilde{\delta}(\cdot)$:

$$\frac{dQ(t)}{dt} = \frac{1}{\Delta n} \left( G(\tilde{s}(t)) Q(t) + D\Sigma(\tilde{s}(t)) \right).$$

(43)

Then for any $t \in [0, T]$, $P(t) = Q(t)Q(t)$.

4. Examples

4.1. Stochastic dynamics

Consider a system with $N = 50$ vehicles and a time horizon of $T = 200$ seconds. Assume a uniform spacing of 0.036 km at time $t = 0$, that is $s(0) = [0.036 \ 0.036 \ 0.036]^\top$. The leader’s speed trajectory is given as

$$v_0(t) = \begin{cases} 
0 \text{ km/hr} & \text{if } t \in [40, 120) \text{ sec} \\
60 \text{ km/hr} & \text{otherwise} 
\end{cases}.$$ 

(44)

The distribution functions, $F_f(\cdot)$, $F_w(\cdot)$, and $F_j(\cdot)$ are Beta distributions supported respectively on $[v_1^{\text{min}}, v_1^{\text{max}}] = [40, 70]$ km/hr, $[w^{\text{min}}, w^{\text{max}}] = [10, 30]$ km/hr, and $[k_1^{\text{min}}, k_1^{\text{max}}] = [110, 170]$ veh/km. Fig. 3 depicts three different sample paths of the stochastic process, which was simulated using Algorithm 1.

Fig. 3: Three sample paths of the same process.
Fig. 4 illustrates convergence of the ensemble averaged process, $s^M(\cdot, \omega)$, to the mean dynamic $\bar{s}(\cdot)$. The differences observed between Fig. 4(b) and Fig. 4(c) can be attributed to numerical diffusion in the $M$ simulations, where a constant and minimum $\Delta t$ was used. In other words the differences are purely numerical.

Fig. 4: Mean dynamics at aggregation levels (a) $s^M(\cdot, \omega)$ with $M = 1$, (b) $s^M(\cdot, \omega)$ with $M = 10$, and (c) $\bar{s}(\cdot)$.

The standard deviation dynamics, $Q(\cdot)$, calculated using (43) are depicted in Fig. 5. What can be observed is a pattern of increasing variability as $n$ increases and with time, particularly in free flow traffic. This is to be expected: in the absence of measurements and due to the randomness in
free-flow speeds, uncertainty about the spacings between vehicles will only
increase with time.

4.2. Variability and data

The purpose of this example is to demonstrate how the covariance cal-
culations (part of the Gaussian approximation) can be utilized in estimating
traffic state. The main idea we wish to convey is how uncertainty about
traffic state behaves in the presence of trajectory data.

We assume a system with \( N \) vehicles (made available, for example, by
a point sensor in the system). We test the impact of increasing trajectory
measurements on the uncertainty about the traffic conditions in the system.
We use a standard Kalman filter, utilizing the derived mean and covariance
dynamics. For the purpose of this example, we assume that data is made
available continuously (or at the same cadence as the numerical scheme used
to predict the mean spacings and covariance). For the sake of completeness,
the Kalman filtering algorithm is given in Algorithm 2 below.

\[ m(t, \omega) = Hs(t, \omega) + \hat{J}^2 \epsilon(t, \omega), \text{ where } \epsilon(t, \omega) \sim \mathcal{N}(0, I) \]

**Algorithm 2: Kalman filter**

**Input:** \( N, T, \Delta t, x_0(\cdot), v_0(\cdot), \{ s_n(0) \}_{n=0}^{N-1} \), measurement equation:

\[ m(t, \omega) = Hs(t, \omega) + \hat{J}^2 \epsilon(t, \omega), \text{ where } \epsilon(t, \omega) \sim \mathcal{N}(0, I) \]

**Iterate:**

1: while \( t \leq T \) do
2: /* Predict state mean */
3: \[ \hat{s}(t+\Delta t|t) = \hat{s}(t|t) + \frac{\Delta t}{\Delta n} D\nabla(\hat{s}(t|t)) \]
4: /* Predict state covariance */
5: \[ Q(t+\Delta t|t) = Q(t|t) + \frac{\Delta t}{\Delta n} \left( G(\hat{s}(t|t))Q(t|t) + D\Sigma^2 \hat{s}(t|t) \right) \]
6: \[ P(t+\Delta t|t) = Q(t+\Delta t|t)Q(t+\Delta t|t) \]
7: /* Residual mean and covariance */
8: \[ r(t+\Delta t) = m(t+\Delta t) - Hs(t+\Delta t|t) \]
9: \[ R(t+\Delta t) = HP(t+\Delta t|t)H^T \]
10: /* Kalman gain */
11: \[ K(t+\Delta t) = P(t+\Delta t|t)H^T R^{-1}(t+\Delta t) \]
12: /* State update */
13: \[ \hat{s}(t+\Delta t|t+\Delta t) = \hat{s}(t+\Delta t|t) + K(t+\Delta t)r(t+\Delta t) \]
14: \[ P(t+\Delta t|t+\Delta t) = (I - K(t+\Delta t)H)P(t+\Delta t|t) \]
15: end while
The ground truth dynamics are depicted in Fig. 6. To see the impact of data availability, we consider the two scenarios depicted in Fig. 7. The estimated spacing dynamics in the 4% and 20% penetration rate cases are shown, respectively, in Fig. 8 and Fig. 9. There is clear improvement in the estimate with the higher penetration rate. In both cases, aggregate measures
of traffic conditions, such as wave dynamics (and speed profiles) are captured with reasonable accuracy.

5. Conclusion

This paper proposes a second-order Gaussian approximation of a stochastic Lagrangian model. The Newell's speed-spacing relation is adopted and the stochasticity is introduced by considering parametric uncertainties (by treating free flow speed, shockwave speed and jam density as random variables).
An ensemble averaged process is derived, which is consistent with traditional first-order Godunov schemes using a mean speed-spacing relation (as numerical flux), not a traditional equilibrium relation. The mean process is shown to converge to a conservation law in Lagrangian coordinates. We then derive the covariance dynamics of the model by applying a Gaussian approximation. One important property of this covariance derivation is that it captures dependence of the covariance matrix on traffic state (namely, spacings) and it is much more tractable than covariance calculations in other data assimilation techniques such as particle filtering or ensemble Kalman filtering. The proposed model was tested with simulated trajectory data. The results show good agreement with ground truth mean dynamics and covariance dynamics, though some numerical diffusion can be observed. In order to demonstrate how this model can be used for traffic state estimation, we applied a standard Kalman filter using the derived mean and covariance dynamics and considering availability of vehicle trajectory data with different penetration rates. Qualitative results show that traffic conditions can be captured with reasonable accuracy.

This study focuses on the derivation of the stochastic model. Future research could be carried out in various directions. For the model itself, we assume deterministic initial and boundary conditions which can be extended to consider uncertainty of initial and boundary conditions. From the application point of view, quantitative validation of the proposed model in terms of queue/delay, wave profile estimation using real data sets (e.g., NGSIM data) is necessary.

References


