The Price of Granularity
And Fractional Finance

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ABSTRACT

The purpose of this paper is to assess the risk premium a fractional financial lognormal (Black-Scholes or BS) process relative to a non-fractional and complete financial pricing model. While fractional Brownian BS models based on the Duncan and Wicks calculus were shown to define a no arbitrage financial model, this paper claim is that this martingale need not be the pricing martingale. There may be many martingales corresponding to a no-arbitrage financial model. In this vein, the intent of this paper are two-fold. On the one hand, provide a definition of the risk premium implied by the discount rate applied to future fractional returns as well as justify this premium compared to a non-fractional financial model. To do so, an insurance rationale for the risk implied by the financial asset volatility is used when defining a no-arbitrage risk neutral probability measure. A “granular risk premium”, is then defined relative to the model granularity. On the other, the paper highlights the effects of a model granularity and its Hurst index on financial risk management. In particular, we argue that such an index is defined by both the granularity underlying the model as well as the effects of mixed models (combining both diffusion and jump processes) on relative range and volatility variations (and thus the measurement of the Hurst index). To present simply the ideas underlying this paper, we price an elementary fractional risk free bond and its risk premium relative to a known spot interest rate. Similarly, the Black-Scholes no arbitrage model is presented in both its non-fractional conventional form and in its fractional framework. The granularity risk premium is then calculated.
1. Introduction

Theoretical Asset Prices are based on financial models that associate securities or other assets future states, say at time “1”, \( S_j(1) \), to “state prices” \( \pi_j \). On the basis of these, a security with known future state preferences, say \( n \) states, has an implied value given by the sum of its cash flows or \( \sum_{j=1}^{n} \pi_j S_j(1), j=1,2...,n \). This means that the current price of a security and its complete future state prices, are known. In this sense, the current price \( S(0) \), implies future state prices cash flows \( \sum_{j=1}^{n} \pi_j S_j(1), j=1,2...,n \) (at time \( t=1 \)). For such models, the assumption is made that there exists an \( n \)-vector of positive state prices \( \pi_j \), such that:

\[
(1) \quad S(0) = \sum_{j=1}^{n} S_j(1) \pi_j \quad \text{or} \quad S(0) \leftrightarrow \{ \pi_j; S_j(1); j=1,2,...n \}
\]

In other words, a current price implies future ones, and vice versa, a future cash flow over all \( n \) state prices implies the current price. In this sense, financial pricing is an inverse problem defined by the hypothesis that future states and their prices are predictable. Such hypotheses lead to the current price to be equal to its future value and therefore, prices then and now are the same. In such circumstances, there are no reasons to trade. These lead to pricing martingales, defining a unique current price. The mathematical properties of such a martingale, defines complete markets. For example, a pricing martingale implies a unique price as well as no-arbitrage, no transaction costs, no information asymmetry, no dominant financial agent, etc. Financial pricing in a complete market model is therefore a mathematical construct based on a shared rationality and an information set that underlies all financial agents’ exchanges and trades. In such contexts, financial pricing models are reference models with respect to which financial traders, buy, sell, bet and value financial assets and exchange at a given price. It assumes further, that all future states and their prices are known and shared, summarized parametrically at a current time (its filtration) and changing over time as new information accrues and its meaning analyzed and interpreted. These elements, under an appropriate martingale, define a probability measure with respect to which an equilibrium price is reached. Financial pricing models are therefore model specific, defined with respect to their assumptions, their known future states, relative to what we know, relative to the information shared by financial agents and relative to the model granularity defining the evolution of pricing models. Fractional models for example, based on time scaling can augment or reduce the number of future states to account for and therefore the information required by financial agents to prices assets. The purpose of this paper is to consider the pricing of a financial asset as a function of its granularity defined by the fractional financial model one may use to price a future price.

2. A Two States Example

Consider a two future states stock pricing model and let the time interval be defined in terms of a fractional parameter \( H \),
\[ \frac{S_H(0)}{B_H(0)} = \frac{S_1((\Delta t)^H)}{B((\Delta t)^H)} P_1^H + \frac{S_2((\Delta t)^H)}{B((\Delta t)^H)} P_2^H \quad \text{with} \quad B((\Delta t)^H) = B_H(0)(1 + R_f^H)^{((\Delta t)^H)} \]

Where \( R_f \) is a constant spot interest rate while \( (\Delta t)^H \) is the maturity date of the bond. When \( H=1 \), we refer to a reference pricing model. As a result, \( S_H(0), B_H(0) \) are stock and bond prices at time \( t=0 \), priced at a current time based on a future (non-fractional) state prices. In this case, both the probability measures \( (p_1^H, p_2^H) \) and the risk free rate \( R_f^H \) are a fraction of the fractions \( (\Delta t)^H \). Over the one period, we have:

\[
S_H(0)(1 + R_f^H)^{((\Delta t)^H)} = E^p_n \left\{ S(\Delta t)^H \right\} \quad \text{where} \quad p_i^H = \left(1 + R_f^H\right)^{-((\Delta t)^H)} \frac{S_H(0) - S_2(\Delta t)^H)}{S_1(\Delta t)^H - S_2((\Delta t)^H)}
\]

defines a probability measure that define the Martingale pricing model:

\[
S_H(0) = \left(1 + R_f^H\right)^{-((\Delta t)^H)} E^p_n \left\{ S(\Delta t)^H \right\}
\]

For a single period, the price \( S_H(0) \) and \( S_1(0) \) are equal for a bond and a stock priced at time \( (\Delta t)^H \).

Over multiple periods, say a final time \( T, \ n_H = T / (\Delta t)^H \), the current price of a stock is a function of its granularity, namely defining a time scale. A Martingale-complete markets model is in this case a reference model defining a unique price resulting from a financial equilibrium based on a common measure of time, predictability of the complete set of future prices and information symmetry (i.e. all agents have the same information and use the same time scale), etc. When any of these assumptions are violated, a financial price might provide an arbitrage opportunity. Both time scaling and their associated pricing models alter the pricing model and in some cases may contribute to both risk assessment and arbitrage pricing. In this case, their difference defines a granularity premium, namely the price advantage that one or the other financial agent’s model has over the other in defining a price. Practically, their difference is due to fractional model being auto-correlated (an assumption often neglected in theoretical pricing models), the financial data an agent uses for constructing a pricing model, the speed of convergence of its underlying probability model and its algorithmic estimate of the current price.

The effect of granularity on pricing models is important for two reasons. First, they define a time measurement scale which necessarily affects the statistical estimate properties based on such measurements. Second, limit calculations such as differential and integral calculations when the time intervals \( \Delta t \to 0 \) and \( (\Delta t)^H \to 0 \) differ in their speed of convergence to infinitesimal and negligible quantities. The latter, tending to zero faster or slower than \( \Delta t \to 0 \), depending on the fractional parameter. Such convergence speed affects a series autocorrelation and thereby its correlation with past events. Granularity may thus alter both the assumptions and the statistical (and risk) properties of the fractional pricing model, providing a departure from the reference complete market models and thereby its price of granularity.

3. Financial Complete Markets and Fractional Finance

A complete pricing model (whether fractional or not) is inverse problem which is defined the unique correspondence between a current a future price. Practically it is defined by an “insurance” adapted

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probability measure, \( f^Q(\cdot) \) that translates future prospects to be in expectation risk free. Their current price is then defined uniquely by the price of a risk free bond whose maturity is the bond maturity. A stock price is then measured by the risk premium of the “model” insurance contract. For example, a risk neutral probability measure, its martingale corresponds at time \( t \) to the following:

\[
S(t) \left(1 + R_j \right)^t = E^Q \left\{ S_j (t+k) \left(1 + R_j \right)^{t+k} \left| \mathcal{F}_k \right. \right\} \quad \text{or} \quad S(t) = \left(1 + R_j \right)^t E^Q \left\{ S_j (t+k) \left| \mathcal{F}_k \right. \right\}
\]

where \( E^Q(\tilde{S}(T)|\mathcal{F}_k) \) is an expectation taken with respect to a model probability measure defining a future Time T prices distribution in state and time. The stock risk premium is thus defined by the probability measure. In continuous time, assuming that future states \( \tilde{S}(T) \) are known and their standardized prices modelled by a distribution (say a normal probability distribution), then at time \( t \),

\[
S(t) = e^{-R_j(T-t)} \int_{-\infty}^{\infty} S(T)dF^Q(S(T)|\mu,\sigma)
\]

with \( \mu,\sigma \), its mean and volatility while \( F^Q(S(T)|\mu,\sigma) \) is the CDF of the probability measure that accounts for the “insurance contract” that has not removed the risk from the financial exchange contract but its risk consequences (and thereby its pricing by a risk free bond).

A fractional financial pricing model in this case, which we may write as follow may assume several forms (and therefore several prices!). A fractional financial pricing models, “alters the measure of time”, “alters the information underlying the model”, “alters the speed of convergence” of a practical time interval to its theoretical continuous time. As a result, the parameters defining the pricing model, their estimates and the information their provide may differ from a reference complete market model. Thus, even if a fractional martingale pricing model is constructed, it need not be a pricing martingale since pricing Martingales are both unique (by definition) and based on informational assumptions commonly shared that based on a rational exchanges between financial agents, leads to a unique pricing-martingale model.

In such cases, model differences arise due to parameters measurements based on the model granularity (whether it is the risk free rate, the asset’s rate of return or its volatility) or the stochastic calculus we apply to manipulate such models. Thus, in real financial time series (say, high-frequency, intraday, day, weekly, etc.) underlying the granularity of financial model may lead to models that differ theoretically and practically.

For example, fractional Brownian financial models (FBM) will necessarily be priced relative to the terms of the bonds and its granularity (i.e. the definition of time intervals at which the bond price is appreciating). Similarly, defining a risk neutral probability measure will entail defining the risk premium that FBM entails, pay it, and remove from the financial pricing model the risk consequences associated to its risk variations. In this case, as with the risk neutral financial pricing martingale, the price of risk is removed from the model and by the law of no-arbitrage, the only remaining risk free rate is the granular risk free rate (and not the spot risk free rate commonly used in non-fractional models). Granularity implies therefore a risk premium (whether positive or negative) which may in practice provide an arbitrage opportunity to traders and investors (even though, all models may be deemed to be no arbitrage financial models). Furthermore, a fractional price (see also, Tapiero, 2015, Tapiero C, Tapiero O. and X. Zhao) defined by a fractional probability distribution—a statistical sum of two distributions, one fat tail
and the other a normal probability distribution, introduces assumptions that violate the premises of complete markets. In this sense, its risk premium equals the premium price implied in departing from a complete market model.

While a large number of approaches are used to construct no-arbitrage financial models (such as portfolio replication, Martingale, Insurance pricing), this paper uses a FBM risk premium modelling approach which in effect is used by the investor to remove future states’ consequences. This premium however, depends on the model granularity and therefore departs from the assumption that typical fractional financial models have made regarding pricing and the no-arbitrage in financial models.

In the real world, financial incompleteness abounds for a broad set of reasons (for example, Baum et al. 1999a,b, Bouchaud and Sornette, 1994, Compte and Renault, 1998, Mandelbrot, 1963, Tsay, 2000). In many cases, however, information and mathematical constructs are used to reduce their incompleteness into completeness. These models imply necessarily, that they can calculate the risk premium for departing from the complete market price. In this sense, the complete market price is an efficient price. Complete markets constructs however hold if all financial agents share information and act according to the same premises that complete market models imply. When this is not the case, fundamental financial models are default models. Financial spreads for example, are a manifestation that point out to financial agents that are not homogenous, that may differ by the information they have, their risk preferences (utilities), their wealth etc. (see for example, Tapiero, 2014 on the Assets Pricing and Economic Inequalities, Quantitative Finance). Further, in practice, financial markets are essentially incomplete, although their premises provide a robust rationale to believe that markets cannot be incomplete “forever” and will therefore revert to an equilibrium state (however flimsy, fast or slow that such a return may be). Thus, although in the real world of finance, markets are incomplete, complete markets provide an important anchor to theoretical finance and a model reference we may use to price the price of its incompleteness.

The intent of the paper is two-fold. On the one hand, define a “model granularity default risk premium”, specifically defined by the use of a fractional-granular approach based on a complete financial market model. On the other, provide a practical interpretation for fractional models and their importance to financial pricing and financial risks management. In particular, fractional pricing models may lead to models of extreme and tail risks (when the Hurst index is appreciably larger than 1) as well as statistically incomplete future pricing probability distributions (i.e. distribution with state space greater than or smaller than 1). Empirical and theoretical studies based on R/S analysis have also shown the effects fractional parameters estimates on risk measurement, risk, “long run, autocorrelated, memory” implied by their covariance growth that indicates a nonlinear volatility growth (defined in some cases by the Hurst index, when it values is greater than \( \frac{1}{2} \), see Tapiero and Vallois, 1996, 1997, Vallois and Tapiero, 1995, 1996a,b, 1997, 2001, 2007, 2008, 2009 as well as Mandelbrot and Van Ness 1968) as well as by the information loss or the information implied (and not available) implied by fractional models.

Although we consider both the price of fractional bonds and the price of stocks with normal (and fractional) rates of returns, the paper provides a relative pricing approach which is based on the prolific existence of fractional no-arbitrage models, that are not financial pricing equilibrium models as they may not be commonly shared, or assume an information which might not be available. In this sense, from a practical pricing point of view, these models may in fact be incomplete.

4. Granularity and its Financial Implications

Continuous time models are based on infinitesimal time and state increments \((dt,dS)\) and on a Riemann calculus. Fractional calculus, generalizes such assumptions by letting time and state increments be \(\left((dt)^\alpha,d^\alpha S\right)\). For example, say that an asset is priced at time intervals’ measure \(dt = 0.01\). If \(0 < \alpha < 1\), and since \(dt < 1\), \((dt)^\alpha > (dt)\), such a record of time has implications to the financial model we use, the estimates and the calculations applied on such a model (for example, a loss of price information, recording of events, etc.) as well as to financial risk management (for example, to missing data jumps in a time series, or the effects of outliers signaled by excessive range statistics or embedded in volatility measurements, etc.). Therefore, a “time clock granularity” defining a fractional parameter \(H\) contributes to a model’s properties and thereby to the statistical and computational properties of such models. Explicitly, let \(S(t)\) be a stock price at a specific time \(t\) and its fractional increment denoting infinitesimal variations in prices in \((dt)^H\).

\[
d^H S = \sum_{i=0}^{\infty} (-1)^i \binom{H}{i} S(t + (H - i)h) - S(t)
\]

The fractional derivative is then defined by \(S^{(H)}(t) = d^H S / (dt)^H\). For example, if a financial price model is defined by increments \((dt,dS)\) while a fractional model is defined by increments \(\left((dt)^H,d^H S\right)\), then of course, the outcomes observed from such models will point out to their differences. However, if \((dt)^H > dt\) then of course, there may be a “loss of model information” due to the fact that intervals are computed at intervals that would be greater than the presumed reference model with time intervals \(dt\). In other words, 2 data sets, one based on estimates of time increments \(dt\) and \((dt)^H\) need not lead to the same results—both theoretically and practically. On the one hand, a fractional (incomplete) financial model may provide arbitrage opportunities, on the other, it imply a financial premium and risk properties (due among other factors to their changing volatility and range as well as to the “controls” provided by the model’s granularity).

As a result, fundamental financial theories based on different granularity, may or may not lead to the same results, nor would they necessarily lead to the same financial conclusions. The time scale of measurement in both financial theory and in practice are thus to be accounted for. While financial pricing models in their simple form are defined by two parameters: the risk free rate, and the underlying price volatility, an equivalent fractional model, provides an opportunity to better define a process characteristics. Models such as ARCH-GARCH that estimate stochastic volatility models have been constructed to better explain the leptokurtic character of rates of returns distributions Baillie 1996, Fox and Taqqu, 1985, Geweke, Porter and Hudak, 1984, Vianoplyby et al., 1994. Other studies have shown
that distributions have tails fatter than the normal distribution and therefore pointing to financial models defaults and thus to their incompleteness. Further, stochastic volatility and fractional Brownian motion models have shown that non-linear volatility growth and non-linear dependence may be observed in fact. In such cases, the assumptions of a linear time and “normal” finance may in practice be doubtful. References to these models and the problems they deal with are numerous.

Considerable theoretical and empirical research (Hurst, 1951, Mandelbrot 1971, 1972, 1997, Mandelbrot and Wallis, 1968, 1969, Mandelbrot and Taqqu, 1979, Mandelbrot and Van Ness, 1968, Lo, 1991, 1997, 1997) have also pointed out that estimates of the fractional parameter $H$ (also called the Hurst index), may be expressed in terms of sample range to volatility ratio estimates (albeit based on infinite sample series and therefore, defining statistical population probability relationships rather than statistical sample relationships) set here and for simplicity to $\left( \frac{R}{s} \right) \propto (T)^H$ where $R/s$ is the sample range to the sample volatility estimate and T is the sample time series on the basis of which estimates of the Hurst index are generated. The fractional parameter $H$ is thus a function of two variability estimates: The sample range which may be due to both the volatility of a diffusion process and jumps of another process, independent or correlated to the diffusion process and the sample standard deviation, both of which are expressing different aspects of variability and therefore their risks and their usefulness to financial risk management. Excessive range relative to volatility statistical estimates may be more indicative of jumps and time series outliers (and therefore important to a financial regulation that would seek to control outlying events). By the same token, stable estimates of the Hurst exponent may point out to an underlying stable diffusion process. When $H = 0.5$, the R/S process underlies a normal process and therefore a time series with noise $\epsilon(t)(\Delta t)^{1/2}$ where $\epsilon(t)$ is a standard normal distribution. Its variance is thus $(\Delta t)^{1/2}$ while when $H > 0.5$ it implies both a greater variance, growing nonlinearly at a rate of $(\Delta t)^{2H} = (\Delta t)^{1/2} \lambda > 0$ as well as potentially a greater outcomes range pointing out to greater instability. This means that the smaller the time interval $(\Delta t)^H$ compared to $(\Delta t)^{1/2}$ the greater the expected ratio $(R/s)$ by an underlying normal process. Further, it may be explained by either a smaller volatility noise or by a greater sample range. And vice versa, it may imply a larger volatility and a smaller range (which is unlikely since volatility estimates are derived from the same data sets that the range is estimated from). From a financial viewpoint, an interpretation of the fractional parameter as that relating to volatility has to be used carefully, as it points out to two interaction and dependent statistics: one underlying the series noise (the volatility) and the other “outliers” or jumps (see also Irwin, 1925, Barnett and Lewis 1994 for a statistical measure of outliers by range statistics). Explicitly, let $S_{(i)}$ be the ith ordered statistic (largest) price in a time interval T (say trades within a day), then $R_{(i)}/\sigma_T = (S_{(i)} - S_{(1)})/\sigma_T$ with $\sigma_T$ the underlying sample volatility may be used as a measure of the sample propensity of an outlier statistic. Neyman and Scott (1971) introduced further the concept of outiler prone and outlier resistant families of distributions which is based on the ratio of two extreme of a sample and the range of that sample. Although the distributions of the range are difficult to compute (Feller, 1951) their distribution and their inverse distribution for random walks Brownian motion and birth and death processes have been computed explicitly (for example, Imhof 1985, Vallois, 1995, 1996, Vallois and Tapiero , 1995-2009 . Further approximate volatility estimates based and range statistics have profusely been used albeit with little success.
Parkinson, 1980 (see also Garman, 1980) using a Feller’s result (1957) provides a volatility estimate given by:

$$\sigma^2 = \frac{1}{Tn(4\ln 2)} \sum_{i=1}^{n} R_i$$

Of course, when a time series underlying noise has a greater volatility, its range statistics will tend to increase as well since both are co-dependent. For example, using an intraday data set or a day data sets, will lead necessarily to different results. For these reasons, theoretical and practical financial models based on discrete time estimates (however small the time interval, since all financial time series are in fact discrete) ought to be far more concerned about the measurement of these series and their interpretation. For references on the range process and process volatility see for example, Baillie, 1996, Beran, 1992, Donkhan et al. 2003, Granger and Joyeux, 1980, Mandelbrot and Taqqu, 1978, Mandelbrot and Van ness, 1986, Mariani et al., 2009).

Below we consider a fractional Bond pricing model to highlight the equivalence and the differences between bond prices of different granularity. These prices will be used subsequently when calculating the price of granular securities and subsequently, the price of granularity.

5. The Risk Free Spot and Fractional Discounts

Consider first and for introductory purposes, a typical financial risk free bond pricing model. Say that the price of a bond is defined in terms of an agreed or legislated spot (continuous time) risk free rate given by \( R_f(t) \) whose price at maturity, say time \( T \), is \( B(T) \). The risk free pricing model is thus:

$$dB(t) = R_f(t)dt, \quad B(T) > 0$$ or equivalently $$d\ln B(t) = R_f(t)dt$$

Its current price is thus: $$0 < B(0) = B(T)\exp\left(-\int_0^T R_f(\tau)d\tau\right).$$ In this case, the bond terminal value and legislated interest rate define the current bond price. Consider instead a fractional bond defined by \( B^H(t) = R_f^H(t)B(t), \ B^H(T) > 0 \) where \( R_f^H \) denotes a risk free discount rate indexed to the fractional parameter. Further, by definition, the fractional derivative the bond is:

$$B^{(H)}(t) = d^{(H)}B(t)/(dt)^{(H)}$$ and \( B^H(T) \) is its fractional price at its maturity at time \( T \). The fractional risk free rate is denoted by \( R_f^{(H)}(t) \) denoting the bond compounding value in fractional time intervals \((dt)^{(H)}\). If \( H = 1 \), \( B_f(T) = B(T) \), while bond prices \( B^H(0) \) and \( B(0) \) need not be the same. In other words, their bonds discount rates need not also be the same. If not, these two prices differ and may provide a granularity arbitrage profit to a bond trader or to a bank borrowing from the Fed with terms transferred to individual borrowers as fractional bonds. Arbitrage profits are then defined in terms of both the fractional parameter and the risk free rates \( R_f^H(t) \) and \( R_f(t) \) or by their spread \([R_f(t) - R_f^H(t)]\) which is a premium time scale for the bond. The proposition below establishes equivalence between spot and fractional risk free rates.

**Proposition 1: Risk Free Bond Prices Equivalence**
Let \( R_f^j(t) \) and \( R_f^H(t) \) be spot and fractional and spot rates of interest at time \( t \), with \( 0 < H < 1 \) and let their current prices be \( \{B_H(0), B_i(0)\} \) with equal payouts at maturity \( T \), \( B_H(T) \equiv B_i(T) \). The price spread of these two bonds is then:

\[
B^H(0) = \frac{B^H(T)}{E_H \left\{ H \frac{d}{dt} \int_0^t (t-\tau)^{H-1} R_f^H(\tau) d\tau \right\} } \quad \text{and} \quad B^i(0) = B^i(T) e^{-\int_0^T R_f^i(\tau) d\tau}
\]

where \( E_H(.,:) = \sum_{i=0}^{\infty} \frac{H^i}{\Gamma(H+i)} \) is a Mittag-Leffler function.

**Proof**

Consider first \( B^H(t) = R_f^H(t)B(t) \) whose solution is given by the Riemann–Liouville function where \( R_f^H(t) \) is the rate of return on a bond paying an interest every time interval \( (dt)^H \). Let, \( 0 < H < 1 \)

\[
\frac{d^{(H)}}{dt^H} \frac{B^H(t)}{B(t)} = R_f^H(t)(dt)^H \quad \text{or} \quad \ln_H \left( \frac{B^H(t)}{B^H(0)} \right) - \ln_H \left( \frac{B^H(0)}{B^H(T)} \right) = H \frac{d}{dt} \int_0^T (t-\tau)^{H-1} R_f^H(\tau) d\tau
\]

And therefore,

\[
B^H(t) = B^H(0) E_H \left\{ \frac{1}{H} \left( \frac{B(t)}{B(0)} \right)^H \right\}
\]

Where \( \ln_H \left( B(t) \right) \) is a fractional logarithmic function, with \( \ln_H \left( B(t) \right) \leftrightarrow B(t) = E_H \left( \ln_H \left( B(t) \right) \right) \).

At \( H=1 \), we have then:

\[
B^i(t) = B^i(0) E_H \left\{ \frac{1}{H} \left( \frac{B(t)}{B(0)} \right)^H \right\} = B^i(0) e^{-\int_0^T R_f^i(\tau) d\tau}
\]

At maturity time \( T \),

\[
B^H(0) = \frac{B^H(T)}{E_H \left\{ H \frac{d}{dt} \int_0^T (t-\tau)^{H-1} R_f^H(\tau) d\tau \right\} } \quad \text{and} \quad B^i(0) = B^i(T) e^{-\int_0^T R_f^i(\tau) d\tau}
\]

And therefore,

\[
\frac{B^H(0)}{B^i(0)} = \frac{B^H(T)}{B^i(T) e^{-\int_0^T R_f^i(\tau) d\tau}} e^{\int_0^T R_f^i(\tau) d\tau} \frac{1}{E_H \left\{ H \frac{d}{dt} \int_0^T (t-\tau)^{H-1} R_f^H(\tau) d\tau \right\} }
\]

If both bonds have the same price and the same payout at maturity, then their interest rates differ. Then:

\[
e^{-\int_0^T R_f^i(\tau) d\tau} = E_H \left\{ H \frac{d}{dt} \int_0^T (t-\tau)^{H-1} R_f^H(\tau) d\tau \right\}
\]

Granularity arbitrage arises when a lender obtains funds on one set of (advantageous) terms and then lends to other agents on the same terms, except by changing the loan granularity. For example, say that the fractional rate of return is a commercial constant, \( \overline{R_f}^H \) set by a bank. Let \( R_f^H(t) \) be a time varying spot.
rate at which rate the lender-banker can borrow from a Central banker. Since these interest rates are constant, say $R_f^H$, then spot and say the granular (commercial) rate are equivalent over a period of time $T$ if:

$$\int_0^T R_f^1(\tau)d\tau = \ln E_H \{H R_f^H T^H\}$$

If both rates are constant, then $R_f^H$ is a solution of the following nonlinear equation, a function of $H$, the fractional index:

$$\frac{dR_f^H}{dH} = \frac{1}{T E_H \{H R_f^H T^H\}} \left( \frac{\partial}{\partial H} \sum_{i=0}^{\infty} \Gamma(H+i) \{H R_f^H T^H\}^{-1} \right) \left( i R_f^H T^H - \frac{\{H R_f^H T^H\}}{\Gamma(H+i)} \frac{\partial \ln \Gamma(H+i)}{\partial H} \right)$$

which can be solved numerically. Similarly, by implicit differentiation, we have:

$$\frac{dR_f^H}{dH} = \frac{\partial}{\partial H} E_H \{H R_f^H T^H\}$$

Granularity and its implications to fixed income pricing and the calculation of fair interest rates settings provides therefore an arbitrage advantage to those capable to calculate the fair terms of exchange (see also Jumarie’s book 2013 as well as the many references to his prior work in the book).

If $H \neq 1$, the terms for no arbitrage may then be defined by solving the equation above. The implication of the fractional parameter in this (deterministic) case is therefore important. If $(dt)^H > dt$ and $H < 1$ we should expect a lender interest rate $R_f^H$ to be greater than that of $R_f^1$. In other words, $R_f^H = R_f^1 + \lambda_H$ where $\lambda_H$ is a fractional compounded premium. For example, say that $dt = 0.001, H = 0.5$ and therefore $(dt)^{0.5} = 0.01$, and let the lender borrow money with an interest rate of 0.005 compounded at time intervals 0.01. If the lender lends to potential borrowers at the same rate, his premium due to compounding effects is (where $B_t(0)$ was set to 1 for simplicity):

$$\ln(1) = 0 = -R_f^1 T + \ln \sum_{k=0}^{\infty} \left( \frac{(R_f^1 + \lambda_H) T^H}{\Gamma(1+kH)} \right)^k$$

$$R_f^1 = \frac{1}{T} \ln \sum_{k=0}^{\infty} \left( \frac{(R_f^1 + \lambda_H) T^H}{\Gamma(1+kH)} \right)^k = 1 - \ln \left( 1 + \frac{0.005 \sqrt{T}}{\Gamma(1.5)} + \frac{(0.005)^2 T}{\Gamma(2)} + \cdots + \frac{(0.005)^n T^n}{\Gamma(1+n/2)} + \cdots \right)$$

Whose solution for $R_f^1$ is necessarily a function of time $T$. If we set, $T=1$, then a quadratic equation results whose solution is:

$$R_f^1 \approx \ln \left( 1 + 2 \frac{0.005}{\sqrt{\pi}} \right) = 0.0056260399$$
and therefore a premium $\lambda_{n,t} = 0.0006260399$ which is 12.52% of the lender investment when charging a continuous time compound interest rate. Inversely, mortgages with monthly, bi-weekly, yearly etc. time payments as well as the time length these borrowing rates are applied to (5, 10 years, etc.) are important considerations in determining the interest rates to be applied to borrowers based on the rates and the terms that lenders are able to borrow money from the Treasury or the Central banks. The inability of borrowers to calculate the interest provides therefore an arbitrage opportunity to lenders.

A direct application to investment in bonds (as well as related problems) by a prospective retiree may be solved as well as a fractional model. The definition and the solution of such a problem is summarized below by the following example.

**Example:** Consider an investor in risk free bonds for a planned retirement date, say $T$, at which time, it plans to save at maturity $B(T)$. The investor’s plan is then defined by how much to invest initially at the spot rate and how much to add in savings $S(t)$ over time. The latter defined by investments added at the time intervals $\left(dt\right)^{H} > dt$. For simplicity, we assume that all investments have a spot risk free rate of return $R_{f}^{t}$ on the outstanding bonds investment. This is reduced to the following fractional model whose solution is given below:

**Proposition 2: Saving for Retirement**

Let a fixed income investment price be defined by:

$$dB(t) = R_{f}B(t)dt + S(t)\left(dt\right)^{H}, B(0) > 0, 0 < H < 1, B(T) = \bar{B}$$

Its solution is a function of the fractional parameter denoting the time interval length between the investor savings $S(\tau)$ at time $0 < \tau < T$ while $\{B(0), B(T)\}$ are the initial investment and the terminal (planned savings) by the retiree:

$$B(T) = B(0)e^{R_{f}^{t}T} + He^{R_{f}^{t}T}E_{H}\left(\Gamma(1 + H)\int_{0}^{T}(T - \tau)^{H}S(\tau)d\tau\right)$$

**Proof:**

The Proof is straightforward. Let $y(t) = B(t)e^{R_{f}^{t}t}$ then,

$$dy(t) = e^{-R_{f}^{t}t}B\frac{dB(t)}{B} - R_{f}B(t)e^{-R_{f}^{t}t}dt = y(t)\frac{dB(t)}{B} - R_{f}y(t)dt \text{ and}$$

$$\frac{dy(t)}{y(t)} = \frac{dB(t)}{B} - R_{f}dt \text{ or } y(t) = B(t)e^{-R_{f}^{t}t} - B(0)$$

which yields the model:

$$\frac{dy(t)}{y(t)} = S(t)\left(dt\right)^{H}, y(0) = B(0) > 0, 0 < H < 1, y(T) = \bar{B}e^{R_{f}^{t}T}$$

And

$$\frac{d^{H}y(t)}{y(t)} = \Gamma(1 + H)S(t)\left(dt\right)^{H}, y(0) = B(0) > 0, 0 < H < 1, y(T) = \bar{B}e^{R_{f}^{t}T}$$

As a result,

$$y(t) - y(0) = \int_{0}^{y(t)}\frac{d^{H}y(t)}{y(t)} = \ln_{H}(y(t)) - \ln_{H}(y(0))$$
And inversely,
\[ E_H \left( \ln y(t) - \ln y(0) \right) = y(t) - y(0) \]

Or
\[ y(t) = E_H \left( \int_0^{t_H} \frac{dH}{\xi(t)} \right) = HE_H \left( \Gamma(1 + H) \int_0^t (t - \tau)^{H-1} S(\tau) d\tau \right) - y(0) \]

And therefore,
\[ B(t)e^{Rt} - B(0) = HE_H \left( \Gamma(1 + H) \int_0^t (t - \tau)^{H-1} S(\tau) d\tau \right) \]

And at time T,
\[ B(T) = B(0)e^{Rt} + HE_H \left( \Gamma(1 + H) \int_0^T (T - \tau)^{H-1} S(\tau) d\tau \right) \]

As indicated above.

Q.E.D.

Thus, given a target saving for time T, \( B(T) \), the investment plan of the investor planning a fixed income investment with a rate of return \( R \) for his retirement at time T is given in terms of the parameters \( \{H, B(0), S(\tau)\} \). Say that \( S(\tau) = \bar{S}_H \) is a constant investment made at time intervals \( (dt)^H \), then:
\[ B(T) = \left( B(0) + HE_H \left( \Gamma(1 + H) \bar{S}_H^{t_H} \right) \right) e^{Rt}, \quad 0 < H < 1 \]

where \( HE_H \left( \Gamma(1 + H) \bar{S}_H^{t_H} \right) \) is the current sum of money the investor would have to pay initially to avoid future payments. Of course, if \( H = 1 \), then:
\[ B(T) = \left( B(0) + e^{\bar{S}_H} \right) e^{Rt}. \]

When \( H > 1 \) we will obtain results that differ from the case \( H < 1 \) treated above, essentially due to the application of the Riemann-Liouville equation. Such a case arises in the next section.

The definition of a risk free spot rate with respect to which risk free investments are made is thus dependent on the model granularity and is an important element to determine the price of complete market models. For fractional models, selecting such a rate is therefore more complex as it ought to be defined in terms of the fractional parameter of the model. These requirements alter necessarily the pricing model under an assumption of complete markets.

Below, we consider for the lognormal pricing model commonly used in class rooms and in pricing options (for example, Black and Scholes 1973 Merton, 1973 and an extensive financial literature extending this model we have referred to). We first derive a complete financial market pricing model which serves as a reference model— with respect to which we value a price derived based on a fractional lognormal model. The result of such an analysis will provide a granularity premium price for the fractional and incomplete financial market model. The model we use differs from related papers in the fractional financial literature that sought to study and implement fractional models for asset prices and the mathematical conditions to their being no-arbitrage models. Instead, we view fractional financial models as computational models that need not be a financial pricing model.
6. The Black-Scholes Model: Complete Markets and Fractional Models

We consider first a lognormal model:

\[
\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t), \quad S(0) = S_0 > 0
\]

where \(W(t)\) is an adapted Brownian Motion process whose variance is \(t\), \(\mu\) and \(\sigma\) are the mean and the volatility of rates of returns. Say that an insurance premium to “remove the risk consequences” from such a stock relative to the price \(R_f\) of a risk free rate bond and given by: \((\mu - R_f)/\sigma\). In this case,

\[
\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t) \equiv R_f dt + \sigma \left[dW(t) - \frac{\mu - R_f}{\sigma} dt\right]
\]

And let the risk free probability measure \(dW^Q(t) = dW(t) - \frac{\mu - R_f}{\sigma} dt\). The complete market financial pricing model is then (under the \(Q\) probability measure):

\[
\frac{dS(t)}{S(t)} = R_f dt + \sigma dW^Q(t), \quad S(0) > 0 \quad \text{and} \quad S(0) = e^{-\kappa t} \mathbb{E}^Q\left(S(t)\right)
\]

The \(Q\) probability measure accounts for the future price of risk by a risk premium and therefore under such a distribution, the stock price is riskless and whose rate of return is a risk free rate. Its price is thus, necessarily, an expectation of the future price discounted at the risk free rate. In this model, the price of volatility risk has already been hedged in the premium \(\mu - R_f\). In other words, the probability measure \(W^Q(t)\) redefines for the investor an economic environment without financial risk consequences. This economic environment does not negate the occurrence of random events but renders an investor oblivious to their consequences and its price which is by definition the price of a risk free bond. The current procedure applied to a fractional financial model will necessarily lead to results that differ from such a pricing model.

Financial pricing models are relative. Just as a lognormal stock model is priced relative to that of another asset (in this case a risk free bond) providing a risk free model for complete markets pricing, a fractional financial pricing model may be measured relative to that complete market pricing model. In other words, a fractional pricing model, is not a pricing model but a relative pricing model as it will, necessarily violate a number of assumption associated to financial complete models.

The introduction of granularity in a financial model has theoretical, computational and risk management implications for financial modeling. Chiridito, 2001a,b, 2003 indicates that granularity introduces a potential for arbitrage and therefore markets are incomplete. Hu and Oksendahl, 2003, provided a solution to a fractional Black and Scholes model to be a pricing martingale and therefore with no formal arbitrage—although a martingale need first to be a pricing model to have no arbitrage. Numerous papers have also provided (based on Duncan, Hu and Pasik-Duncan, 2000, seminal paper on fractional stochastic calculus), numerous financial pricing applications. Elliott and Vander Hoek, 2003, provide a general fractional white-noise theory based on the Wicks-Ito-Skorohod (WIS) calculus applied to finance (see also Nualart and Taqqu, 2006). Jumarie, 2005a,b,c and 2008, 2009, applies and modifies the Riemann-Liouville derivative and fractional Taylor series to non-differentiable functions (Jumarie, 2006, Liouville, 1832). Financial pricing models are however “statistical” in the sense that future are defined by
constructing a complete probability distribution that defines future state prices. Their fractional transformation thus ought to result as well as fractional probability distribution (while the Liouville kernel is a functional transformation). In other words, letting \( K_H(t) \) be a fractional kernel, defining a fractional Brownian Motion, by:

\[
W_H(t) = C_H \int_0^t K_H(t-\tau) dW(\tau)
\]

Then, for the fractional \( W_H(t) \) to define a fractional "noise" in a financial pricing model, we require that \( W_H(t) \) be defined by a complete probability distribution. Further since, this expression essentially provides a fractional time, it is the volatility (rather than the distribution) that is being fractioned. By contrast, consider an exponential probability distribution \( f(t) = \mu e^{-\mu t} \) which in a small time interval is given by:

\[
f(t + \Delta t) = \mu e^{-\mu (t+\Delta t)} \approx f(t) \left[ 1 - \mu \Delta t + \frac{1}{2} (\mu \Delta t)^2 + \ldots \right]
\]

while its hazard rate is \( f(\Delta t) / [1 - F(\Delta t)] = \mu \). In this case, it is the distribution itself which is fractioned. In fact, for a fractional exponential probability distribution, we can expect (an exact solution may be derived easily however, Tapiero et al., 2015):

\[
f\left(t + (\Delta t)^H\right) = \mu e^{-\mu t (\Delta t)^{2H}} \approx f(t) \left[ 1 - \mu (\Delta t)^H + \frac{1}{2} \left( \mu^2 (\Delta t)^{2H} \right) - \frac{1}{3!} \left( \mu^3 (\Delta t)^{3H} \right) + \ldots \right]
\]

In this sense, a Fractional Brownian Motion (FBM) is fractional only in the sense of its volatility since only time is fractioned and not the probability distribution.

In a FBM, \( W_H(t) \), random events \( dW_H(t) \) in a time interval \( (dt)^H \) recur and are compounded in time increments \( (dt)^H \). If \( 0 < H < 1 \) then \( (dt)^H > dt \) and therefore, its variance is \( (dt)^{2H} > dt \), a variance larger than that of a Brownian motion increments \( dW(t) \) when \( H < \frac{1}{2} \) and smaller than \( dt \) if \( H > \frac{1}{2} \). Evidence of volatility “stabilization” associated to \( H < 1/2 \) for example is given by Otway 1995, for an anti-persistent economic time series based on records of the Florentine proveditori degli cambiatori (black Tulip trades in the Middle Ages). If \( H > \frac{1}{2} \), the time interval \( (dt)^H < dt \) points to financial models representative of high frequency trades for example. However in FBM its variance is necessarily \( \sigma^2 (dt)^{2H} \) (rather than \( dt \)), \( 1 < 2H < 2 \) and therefore, it would seem that if an underlying “noise” is a FBM, and \( H > 1/2 \), then variance growth is “persistent”, growing nonlinearly. In other words, the larger the Hurst index, and assuming that the underlying data is normally distributed the fractional time would account High and Ultra-High frequency financial data (which is not probably the case). Fractional stochastic models are therefore computationally specific, depending on both the stochastic calculus we use to integrate stochastic integrals and our interpretation of fractional model parameters and the assumptions we make regarding times series probability distributions. Further, a financial stochastic differential equations combining both a drift and random (Fractional Brownian Motion Noise) introduces
as well a mixture of time scales, one of order \((dt)\) representing a “long run” mean trend, and the other a persistent variance proportional to \((dt)^{2H}, 1 < 2H < 2\).

Generally, a correspondence between a fractional and a standard Brownian motion is given by a kernel functional equation \( K_H(t, \tau) \) which is a function of the fractional parameter:

\[
W_H(t) = \int_0^t K_H(t-\tau) dW(\tau)
\]

Using the Riemann-Liouville function for a fractional parameter \(0 < H < 1\) a correspondence to a standard Brownian motion is for example (Mandelbrot and Van Ness, 1965) a kernel:

\[
K_H(t-\tau) = \frac{(t-\tau)^{(H-1)}}{\Gamma(H+1)}
\]

and therefore,

\[
W_H(t) = \frac{1}{\Gamma(2H)} \int_0^t (t-\tau)^{(2H-1)} dW(\tau)
\]

with \(H = 1/2\) and Brownian Motion \(W(t)\).

The fractional kernel \(K_H(t, \tau)\) is a stochastic integral that define the FBM \(\{W_H(t), t \geq 0\}, \forall t \in \mathbb{R}_+\) with: \(\Pr\{W_H(0) = 0\} = 1\) as a measurable random variable such that it has a null mean and a covariance given by:

\[
E\{W_H(t)W_H(\tau)\} = \sigma_H^2 C_H \{t^{2H} + \tau^{2H} - |t-\tau|^{2H}\}
\]

and a variance of the order \(C_H t^{2H}\) growing faster or slower than a linear time due to its Hurst (fractional index) as indicated above. This behavior of the variance has led to the naming of such models based on their time variant co-variation. In particular names such as long range dependence, strong dependence, or a stationary process with slowly decaying or long range correlation are given to such processes, in particular when \(2H > 1\) (e.g. Bloomberg, as a “chaos coefficient” and more formally, a self-similarity index).

Over a time interval \(\Delta \tau\), the probability distribution of the fractional Brownian motion is of course normally distribution with a zero mean and a variance which is a function of the fractional index. An intuitive proof is as follows. Let \(\Delta W(\tau) = W(\tau) - W(\tau - \Delta \tau)\) which are normally distributed random variables with mean zero and variance \(\Delta \tau\). As a result, approximately, using the Liouville kernel, we have:

\[
W_H(t) = 2H \sum_{i=1}^{n}[\Delta \tau_i^{2H-1} \Delta (W(\Delta \tau))] , n = \frac{t}{\Delta \tau}, \tau = i \Delta \tau
\]

In continuous time the fractional volatility of a Brownian Motion yields:

\[
W_H(t) = 2H \int_0^t [t-\tau]^{2H-1} dW(\tau)
\]

Consider instead a complete market financial pricing model,

\[
\frac{dS(t)}{S(t)} = R_f dt + \sigma dW^Q(t), S(0) = S_0 > 0 \quad \text{with} \quad W^Q(t) = W(t) - \frac{\mu - R_f}{\sigma} t
\]

While its fractional equivalent model is:

\[
\frac{d^H S(t)}{S(t)} = R_f^H (dt)^H + \sigma dW_H^Q(t), S(0) = S_0 > 0
\]
where $W^Q_H(t)$ is a fractional measure of the BM probability measure $W^Q(t)$ while $R_f^H$ and $\sigma_H$ are two parameters associated to the FBM process. As indicated in the previous section, for constant risk free spot rate, the fractional risk free rate is given by a solution of:

$$R_f = \frac{1}{t} \ln E_H^Q \left\{ H \tilde{R}^H f(t) \right\}$$

and therefore $R_f^H(t) = \Phi \left( \tilde{R}_f, t \right)$ and $\tilde{R}_f = \int_0^t R_f^H(t) \, dt \Phi \left( \tilde{R}_f, t \right)$.

As a result, for both constant fractional and spot rates, we have a polynomial function for the fractional rate given by: $R_f = \frac{1}{t} \ln E_H^Q \left\{ H \tilde{R}^H f(t) \right\}$. Generally, we have:

As a result, we have:

$$d^H S(t) = R_f^H(t)(dt)^H + \sigma_H^H dW^Q_H(t), S(0) = S_0 > 0$$

Consider the solution of this stochastic differential equation, in two parts by setting, $S_H(t) = S_1(t)S_2(t)$ where

$$d^H S_1(t) = R_f^H(t)(dt)^H, S_1(0) > 0 \quad \text{and} \quad d^H S_2(t) = \sigma_H^H dW^Q_H(t), S_2(0) > 0$$

The first part is:

$$d \ln S_1(t) - d \ln S_1(0) = R_f^H(t)(dt)^H \quad \text{and therefore} \quad S_1^H(t) = S_1(0) E_H^Q \left[ \int_0^t \tilde{R}_f^H(\tau) d\tau \right]$$

or,

$$S_1(0) = E_H^Q \left[ \int_0^t \tilde{R}_f^H(\tau) d\tau \right]^{-1} S_1^H(t)$$

where $e^{\tilde{R}^H_t} = E_H^Q \left\{ H \tilde{R}^H f(t) \right\}$ or $\tilde{R}_f^H = (H \tilde{R}^H f)^{-1} \left( e^{\tilde{R}^H t} \right) = \Phi \left( \tilde{R}_f, H, t \right)$ which can be calculated numerically.

While its second part is:

$$d^H S_2(t) = \sigma_H^H dW^Q_H(t), S(0) = S_0 > 0 \quad \text{Or} \quad d^H \ln S_2 = \left( -\frac{1}{2} (\sigma_H^H)^2 (dt)^H \right) + \sigma_H^H dW(t), \quad 2H > 1$$

As a result,

$$d^H S_{2,2}(t) = d \ln \left( S_{2,2}(t) \right) - d \ln \left( S_{2,2}(0) \right) = (\sigma_H^H) \int_0^t dW^Q_H(t) \quad \text{or} \quad S_{2,2}^H(t) = S_{2,2}^H(0) E_H^Q \left( \sigma_H^H W^Q_H(t) \right)$$

Similarly,

$$\ln S_{2,1}^H(t) - \ln S_{2,1}^H(0) = \left( -\frac{1}{2} (\sigma_H^H)^2 2H \int_0^t (dt)^H \right) \quad \text{or} \quad S_{2,1}^H(t) = S_{2,1}^H(0) E_{2H}^Q \left( -\frac{1}{2} (\sigma_H^H)^2 2H (t)^{2H} \right)$$

In this case,

$$S_H(t) = S_H(0) E_H^Q \left[ \int_0^t \tilde{R}_f^H(\tau) d\tau \right] E_{2H}^Q \left[ -\frac{1}{2} (\sigma_H^H)^2 2H (t)^{2H} \right] E_H^Q \left( \sigma_H^H W^Q_H(t) \right)$$

And therefore the price of a fractional volatility Brownian Motion lognormal model is given by calculating the expectation of:
\[ S_H(0) = S_H(t) \left\{ E_H \left( \int_0^t R_H^t (\tau)d\tau \right) E_{2H} \left( -\frac{1}{2} (\sigma_H)^2 2H (t)^{2H} \right) E_{H} \left( \sigma_H W_H^0 (t) \right) \right\}^{-1} \]

And the fractional price is:

\[ S_H(0) = E_H^0 (S_H(t)) \left\{ E_H \left( \int_0^t R_H^t (\tau)d\tau \right) E_{2H} \left( -\frac{1}{2} (\sigma_H)^2 2H (t)^{2H} \right) E_{H} \left( \sigma_H W_H^0 (t) \right) \right\}^{-1} \]

Where expectation is taken with respect the fractional volatility normal probability distribution whose moments are given by its moment generating function as will be shown subsequently.

This price is not a complete market model however, except when \( H=1 \), for a linear time in \( (t)^H \) and \( H=1/2 \), for linear time in the variance \( (t)^{2H} \). In this, case:

\[ S_H(t) = S(0)e^{\sigma (\tau + \sigma W(t) - \frac{1}{2}\sigma^2_t \tau) \frac{1}{2}} \quad \text{or} \quad S(0) = E^0 (S(t)) \]

While the financial price of granularity measures the price derived from a fractional model and the references complete market model, or a \( S_H(0) - S_H(0) \). This price is a function of the time scale \( t^H \) expressing the mean returns trend as well as its volatility given by \( \sigma_H (dt)^H \) whose variance is \( \sigma_H^2 (dt)^{2H} \) in which case, the price of granularity (when \( 2H>1 \)) will necessarily increase due to its volatility growth and tail risk (due to the fractional Brownian Motion). Both the effects of volatility and tail risks can be verified by considering the fractional moment generating function of the fractional distribution.

7. The Fractional Volatility Brownian Motion

Fractional Brownian Motion in in fact a normal probability distribution with fractional volatility. In other words, it fractionalizes the time defining its underlying volatility and not its distribution as was the introduced for the fractional probability distribution. Consider the (deterministic) Liouville kernel transformation:

\[ W_H(t) = C_H \int K_H(t-\tau)dW(\tau) \]

where \( dW(\tau) \) assumes values in \( (-\infty, +\infty) \) and is the Brownian Motion. The Liouville kernel adapted to the infinite limits of the integrals are then defined over the positive and negative domains of a distribution whose variance is time linear:

\[ W_H(t) = C_H \int \left\{ \left( \text{Max}(t-\tau,0) \right)^{\frac{1}{2}} - \left( \text{Max}(-\tau,0) \right)^{\frac{1}{2}} \right\} dW(\tau) \]

The constant \( C_H \) may be calculated by standardizing its value. Explicitly,

\[ C_H = \Gamma \left( H + \frac{1}{2} \right) \left[ \left( \int_0^\infty \left( (1+\tau)^{\frac{1}{2}} - (\tau)^{\frac{1}{2}} \right) d\tau + \frac{1}{2H} \right) \right]^{-\frac{1}{2}} \]

When \( H=1/2 \), then \( C_H = 1 \). Explicit calculations are given by:

\[ W^{1/2}(t) = C_{1/2} \int_0^t \left( t-\tau \right)^{\frac{1}{2}} dW(\tau) = C_{1/2} \int_0^t dW(\tau) = C_{1/2} W(t) \quad \text{and} \quad C_{1/2} = 1 \]
The fractional volatility of the Brownian Motion variance can be calculated as well:

\[ E[W_H(t)^2] = (C_H)^2 E \int \left[ \left( (t-\tau)_{+}^{\frac{1}{2}} - (-\tau)_{+}^{\frac{1}{2}} \right)^2 \right] dW(\tau)^2 = (C_H)^2 \int \left[ \left( (t-\tau)_{+}^{\frac{1}{2}} - (-\tau)_{+}^{\frac{1}{2}} \right)^2 \right] d\tau \]

Replace \( \tau = u \), then:

\[ E[W_H(t)^2] = (C_H)^2 \int E \left[ \left( (t-u)_{+}^{\frac{1}{2}} - (-u)_{+}^{\frac{1}{2}} \right)^2 \right] du = (C_H)^2 \int E[W_H(1)^2] = \]

By the same token, one may show that the variance can be calculated as follows. First we set:

\[ E[W_H(t) - W_H(\tau)]^2 = |t-\tau|^{2H} E[W_H(1)^2] \]

And therefore,

\[ E[W_H(t)W_H(\tau)] = \frac{1}{2} \left( t^{2H} + \tau^{2H} - |t-\tau|^{2H} \right) \]

For \( H > 1/2 \), we shall note that applying Liouville to the case \( 1 < 2H < 1 \):

\[ \int_0^t \left( f^2(\tau) \right) (d\tau)^{2H} = 2H(2H-1) \int_0^t \left( f^2(\tau) \right) |\tau-u|^{2H-2} dud\tau \]

when the double integration is due to the fractional parameter \( 2H > 1 \):

\[ \int_0^t \left( f^2(\tau) \right) (d\tau)^{2H} = 2H(2H-1) \int_0^t \left( (\tau-u)^{(2H-2)} \right) f(\tau) du d\tau = 2H \int_0^t \tau^{2H-1} f(\tau) d\tau \]

For example, for \( f(\tau) \) a constant, we have as expected \( \int_0^t (d\tau)^{2H} = (t_1)^{2H} - (t_0)^{2H} \), the fractional variance of the Brownian Motion. Generally,

\[ \int_0^t \tau^{2\beta} (d\tau)^{2H} = \frac{\Gamma(1+2\beta) \Gamma(2H)}{\Gamma(2\beta+2H)} \int_0^t \tau^{2H+2\beta-1} d\tau = \frac{\Gamma(1+2\beta) \Gamma(2H)}{\Gamma(1+2\beta+2H)} t_1^{2H+2\beta} \]

When \( H = 1/2 \)

\[ \int_0^t \tau^{2\beta} (d\tau) = \frac{t_1^{1+2\beta}}{1+2\beta} \]

8. The Fractional Moment Generating Function

The Moment Generating Function of the Normal distribution with variance proportional to time \( t \) is easily checked to be: \( M(q|t) = E(e^{q\sigma W(t)}) = e^{tq^2} \), \( \lambda = \frac{1}{2} q^2 \sigma^2 \). The time parameter \( t \) accounts for the instant at which the probability distribution standard deviation increasing linearly while that of a fractional is defined by the convolution with a kernel \( \Phi(H|t) \), or \( M_H(q|t) = \Phi(H|t)(M(q|t)) \). Note that since: \( W_H(t) = \int_0^t \frac{1}{M_H(t-\tau)dW(t) \text{ in MGF, we have for } 2H > 1:} \]
\[ M_H(q|t) = t^{2H-1} \sum_{i=0}^{\infty} \left( \frac{q^2 \sigma^2 t^2}{i!} \right)^i \frac{\Gamma(2H-1)\Gamma(2i+1)}{\Gamma(2H-1+2i)} \]

As is proved below.

\[ M_H(q|t) = \int_0^t \int_0^t (u-\tau)^{2H-2} e^{\lambda \tau} d\tau du = \int_0^t \int_0^t \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} (u-\tau)^{2H-2} \tau^i d\tau du \]

And therefore,

\[ M_H(q|t) = \int_0^t \frac{d}{du} \int_0^u (u-\tau)^{2H-2} e^{\lambda \tau} d\tau du = \int_0^t \frac{d}{du} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} (u-\tau)^{2H-2} \tau^i d\tau du \]

\[ t^{2H-1} \sum_{i=0}^{\infty} \frac{\left( \lambda t^2 \right)^i}{i!} \frac{\Gamma(2H-1)\Gamma(2i+1)}{\Gamma(2H-1+2i)} = t^{2H-1} \sum_{i=0}^{\infty} \frac{\left( \frac{1}{2} q^2 \sigma^2 t^2 \right)^i}{i!} \frac{\Gamma(2H-1)\Gamma(2i+1)}{\Gamma(2H-1+2i)} \]

If \( H=1/2 \), we obtain as expected:

\[ M_{1/2}(q|t) = \sum_{i=0}^{\infty} \frac{\left( \lambda t^2 \right)^i}{i!} = e^{\lambda t^2} = e^{\frac{1}{2} q^2 \sigma^2 t^2} \]

In this case,

\[ M_H(q|t) = t^{2H-1} \left\{ 1 + q^2 \sigma^2 t^2 \frac{q^2 \sigma^2 t^2}{2H(2H-1)} + \frac{\left( \frac{1}{2} q^2 \sigma^2 t^2 \right)^2}{2} \frac{\Gamma(2H-1)\Gamma(5)}{\Gamma(2H+3)} + \ldots \right\} \]

And

\[ \frac{\partial M_H(0|t)}{\partial q} M_H(q|t) = t^{2H-1} \left\{ \frac{q \sigma^2 t^2}{H(2H-1)} + \ldots \right\} = 0 \]

\[ \frac{\partial^2 M_H(0|t)}{\partial q^2} = \frac{\sigma^2 t^{2H}}{H(2H-1)} \]

And higher order moments may similarly be calculated.

Fractional models have thus important implications to assets pricing and to the many paradigms used in modern financial economics as well as for financial risk management (since volatility and tail risk increase). Further, it has implications to financial statistics, providing ample opportunities to differentiate between standard and fractional statistics as well as providing granularity as an essential parameter in the control of financial statistics. Problems also arise in the pricing of options and futures since the class of models used are incompatible with fractional models when these models do not define a pricing martingale or are based on a “no-arbitrage” but non-pricing martingale. Traditional tests of the capital asset pricing model and the APT (Arbitrage Pricing Theory) are no longer valid since the usual forms of statistical inference do not apply to time series exhibiting such fractional dependence. Tests of an « efficient » markets hypothesis depend therefore precariously on the presence or absence of fractional models. Rather, these models assume implicitly a fractional time interval which is set to be the reference unit time interval on the basis of which the pricing martingale is defined.

Conclusion and Discussion
Application of fractional calculus to financial models abound. Recently Rostek, and Schöbel, (2013) provided a note on how to use and interpret fractional Brownian motion models for financial modeling. Prior and related studies include Mandelbrot and van Ness, 1968., Cheridito, P., 2001, Elliott, R.J., Van Der Hoek, J., 2003, Hu, and Øksendal, B.2003, Dung 2013). Bjork, T., Hult, H., 2005 uses the Wick product to derive a fractional approach to the Black Scholes model. Such an approach however does not recognize the economic mechanism that renders a stochastic model, a stochastic “risk free” financial pricing model (since the risk consequences of the underlying price have been accounted for by an appropriate risk premium). Other approaches based on fractional models and their calculus are provided by Gu, H., Liang, J., Zhang, Y., 2012, Jumarie, G., 2005, Jumarie, G, 2008, Meng, and Wang, 2010, Rogers, 1997, Rostek, 2000, Sottinen, and Valkeila, E., 200. These are mostly generalization of financial models to include the granularity implied in fractional models. This paper unlike previous approaches provided a practical clarification of a fractional pricing models based on its granularity. Financial models in general combine both stochastic and non-stochastic elements and their pricing consist therefore in how we account for their implied risk. For example, the risk premium paid for an asset’ volatility allows one to remove the consequences of the asset risk and thereby, price such an asset using a risk neutral probability measure.

This paper particularity is that it recognizes explicitly this risk and constructed a fractional model consistent with the price (the spread) of this risk. In this paper framework, unlike previous financial fractional models, a price is obtained relative to a complete market model. In other words, while a fractional model may be incomplete, it still has a price which is measured relative to a financial model which is complete. We chose to interpret this price as a price of granularity due to the model it being fractional. A fractional model may however be also a model of complete markets, in which case, all other financial fractional (or not) models may be priced relative to that complete market financial model. This presumption is reinforced by the fact that all pricing are relative with the risk neutral pricing model based on the price of an asset relative to that of a risk free bond asset. Not least of course are numerous papers by Mandelbrot 1971, on the limits of arbitrage, in 1972, on the covariance and R/S analysis, with Taqqu, 1979, on Robust R/S analysis of long run serial correlation, with Van Ness, 1968, on Fractional Brownian motion, with Wallis, 1968, on Noah, Joseph effects as well as in 1969 on computer experiments with fractional noises,

Although fractional models assume that model events time intervals are constant (although parametrized by a fractional parameter), extension based on random time intervals (jumps) could be considered. Typically, Poisson Jump models assume that the time interval between events is exponential in which case, granularity over a number of integer time intervals has a Gamma distribution while over a non-integer it turns up to be of the fractional Poisson type. In this case, events (whether stochastic or not) do occur randomly, providing a far greater wealth of potential models that account for their real occurrence. For example, interest rate models do not vary continuously and are changed following some (often intractable) random and discrete time patterns. Continuous time models are of course a convenience that has a price.

Applications of fractional models to finance and related problems abound. Cajueiro and Tabak, 2005 tested for time-varying long-range dependence of volatility for emerging markets. In 2007, they test the hypothesis that crude oil markets are weakly efficient over time. To do so a time-varying long-range
dependence in prices and volatility model was considered. Subsequently, in 2008, they tested for time-varying long-range dependence in real states equity returns. Zeynel Abidin Ozdemir, 2009, uses a long run memory model to test the linkage between international stock markets. Willinger, et al. Taqqu, 1999, similarly studied stock market prices and long-range dependence, Xiao, et al 2010a,b,c consider in a series of three papers problems spanning currency options in a fractional Brownian motion with jumps as well as European options with transaction costs using a fractional Brownian motion as well as a mixed fractional system (to obtain a no-arbitrage model). For earlier paper see for example Fung et al, 1994, examining the dependency in intra-day stock index futures, Fung and Lo, 1993, testing for long run memory in interest rate futures, Grange 1980, on Long memory relationships and the aggregation of dynamic models, Green and Fielitz, 1977, on long term dependence in common stock returns and in 1980, on long term dependence and least squares regression in investment analysis, Helms, Kaen and Rosenman, 1984, on Memory in commodity futures contracts, Lo 1991 on long term memory in stock market prices, and in 1997, on Fat tails, long memory and the stock market (a review paper since the 1960's).

Ramirez et al. 2008, consider a time varying fractional exponent Hurst index for US stock markets; Kumar, and Nivedita, 2009, similarly consider the Multifractal properties of the Indian financial market, Gu, et al, 2012 consider a Time-changed geometric fractional Brownian motion and option pricing with transaction costs. By the same token, Gray, et al provide a generalized fractional processes, while Osler, 1971 provides generalized Taylor’s series for fractional derivatives and outlines as well some of their applications.

The essential contribution of the paper however is meant to provide a fractional finance approach based on the fundamental principles we use in constructing financial complete market models. Further, although a financial model with a given granularity may theoretically be complete, in fact it might not define a financial pricing complete market model. In this case, it may be called incomplete. Incompleteness is then embedded in our real observation of time series and behaviors that negate the assumptions of need not negate complete markets. In such situations, incompleteness arise because financial agents do not trade or exchange based on such models. The price of incompleteness is then relative to a complete market with agreed on granularity.

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