Markets, Profits, Capital, Leverage and Return*

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Abstract

The theory of two price markets of Cherny and Madan (2010) yields closed forms for bid and ask prices. Defining profits as the difference between the mid quote and the risk neutral expectation and capital as difference between the ask and the bid price one obtains precise expressions for these entities and thereby also returns. New expressions are developed for the bid and ask prices in terms of the sensitivity of the inverse distribution function to the quantile level. The latter turns out to be a measure of risk exposure at the quantile level. The theory is illustrated on unhedged exposures in the Black Merton Scholes model, followed by variance swaps and call options for variance gamma underliers. It is argued that markets should economize capital and furthermore the maximization of expected utility may involve an uneconomic use of capital. We further observe that stock positions should be revised downwards from zero delta in left skewed markets in response to the target gamma when minimizing capital commitments.

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1 Introduction

The classical theory of financial markets does not explicitly define the level of cash reserves to be maintained in support of a trade, or what is the up-front profit that may be taken on the trade. Consequently, both the leverage provided and the rate of return are unclear. These are the entities that we focus on here and we shall provide at a conceptual level explicit procedures for evaluating capital, profits, leverage and returns on trades in markets. For a specialized and illustrative context we shall also provide closed form formulas for all four. In this respect, this paper attempts to fill a gap in the theory of financial markets and the management of financial risks in the economy. Recognizing the absence of arbitrage one has all trades exposed to loss that must therefore be backed by capital. Additionally such capital commitments occur in anticipation of positive or negative returns for risks taken up or shed. The questions addressed here are important for risk control in the developing and established markets of structured finance and guidance from financial theory on these matters is called for.

As a first step we begin with the slightest generalization of the classical theory of financial markets that allows one to offer an initial answer to the questions we raise. In the classical complete markets theory for financial markets (Dybvig and Ross (2003)) the law of one price prevails and no arbitrage principles pin down a unique risk neutral or pricing measure. Economic agents can trade any amount of any financial product at the going market price that is its cost of replication. No cash reserves are needed as we have full replication with no residual risk to be held. Capital supporting trades are zero. Competition drops profits to zero as all products are provided at cost of replication. Rates of return and leverage are thereby undefined.

Our departure from this classical theory comes in altering just one assumption in the definition of the financial market. All economic agents can still trade with the market at any desired amount, but now the terms of trade depend on the direction of trade. We therefore merely replace the law of one price by the law of two prices with a different price for buying from the market to that for selling to the market. Further and more drastic departures from the classical model are possible. For example one let prices depend on the size of the trade as well as the direction and one may consider game theoretic approaches as opposed to a passive modeling of markets, with the games becoming stochastic as well. Tentatively we argue that our simple, single period, static and passive departure is itself rich enough with much yet to be understood.

We emphasize at the start, that we shall model markets directly with their own rules rooted in competition. In this regard we distinguish the modeling of markets from the modeling of objectives for agents in the economy. The classical Arrow and Debreu (1954) theory recognized this and the explicit recognition was apparent in the classical proofs for the existence of an equilibrium in which a special role is created for the Walrasian auctioneer as the only non-maximizing agent in the economy, focused entirely on getting markets to clear. Yes, markets are made up of people, but by virtue of the atomicity of individuals in markets,
the modeling of markets diverges from the models for individual agents. In this regard we foretell that in our perspective, even though agents may maximize expected utility, this is not an appropriate objective for the market that is not a person, or an aggregate of persons. We shall model markets as competitively economizing the commitment of capital, but first we have to define the latter.

Our model for two price markets follows Cherny and Madan (2009, 2010) and yields closed forms for bid and ask prices in terms of concave distortions of distribution functions. In this paper we shall also obtain new expressions for these prices in terms of the derivative of the inverse of the distribution function. This derivative is seen to play a critical role measuring the risk exposure of a trade at a quantile level and both profits and capital are obtained as integrals over quantiles of charges for exposures measured this way. We shall describe the profit and capital charge at each quantile and explain the critical role played by the quantile sensitivity of the inverse distribution function. More generally, at some computational and modeling cost one may condition the quantile exposures to levels of factor risks and then we would have to integrate over the joint distribution of factor levels or work with factor specific definitions for profit and capital (Christopherson, Ferson and Turner (1999)).

We go on to formulate our definition of up-front profit as the mid price of bid and ask less the cost of replication given by a risk neutral expectation. The capital charge for the issuance of a state contingent liability is defined as the difference between the ask and bid prices. Thus we obtain closed forms for both profit and capital and hence the leverage and rate of return on the trade. We also show that from the viewpoint of the market the hedging strategy should be chosen to minimize capital commitments. This gives us a new market based criterion for the choice of hedging strategies, distinct from the interests of individuals. The latter may be seen as maximizing expected utility or some other preference based criterion, but we model markets as seeking to economize on the commitment of reserve capital.

Four applications illustrate the new methodologies. First we consider unhedged positions in options in the classical Black and Scholes (1973) and Merton (1973) model. The second considers variance swaps when the underlier is a variance gamma process (Madan and Seneta (1990), Madan, Carr and Chang (1998)). The variance swap is a good contract to study in this context for here we do not have a perfect hedge, yet the classical hedge is a good one (Carr and Lee (2010)). We exhibit the quality of the hedge by demonstrating how much reduction is attained in capital requirements by employing the classical hedge. Furthermore, one may improve upon the classical hedge to reduce capital even further. This last step is of course model dependent.

Our third application considers the hedging of a call option for a variance gamma underlier. We show that on a one year contract one may reduce capital commitments by quarterly hedging and hedging monthly does not make a significant improvement over this. We also show that the maximization of expected utility may result in a significantly higher capital commitment when compared to implementing market objectives of capital reduction.

For our fourth application we ask in the joint context of a negatively skewed
risk neutral return distributions and the inability to locally cover all possible market moves whether the objective of minimizing capital commitments suggests a delta hedging policy that is sensitive to the gamma of a position as well as its delta. We analyze this question in the context of the variance gamma model by choosing positions in stock to minimize capital given residual risk exposure. We then regress the optimal capital minimizing stock positions on the delta and gamma of the target cash flow. We observe that deltas should be reduced in response to the gamma and more so when the volatility surface has a larger skew.

The outline of the rest of the paper is as follows. Section 2 briefly presents the theory of two price markets as set out in Cherny and Madan (2010) along with the closed forms for bid and ask prices using concave distortions. Section 3 presents the results on expressing these prices in terms of the derivative of the inverse distribution function and the role played by this function as a measure of risk exposure. Section 4 develops the rationale for profits and capital and the closed forms for these in terms of exposure sensitivities as measured by the slope of the inverse distribution function. Section 5 relates our measures of capital to the leverage being provided. The remaining sections report on the applications with Section 6 presenting the results for unhedged options under geometric Brownian motion. Section 7 takes up the variance swap contract under the variance gamma model for the underlier. Section 8 reports on hedging call options with a view to minimizing capital for a variance gamma underlier. Section 9 presents our analysis of delta hedging in left skewed markets. Section 10 concludes.

2 Two Price Markets

This section presents the theory of two price markets and the closed form formulas for bid and ask prices using concave distortions as developed in Cherny and Madan (2010). In this approach the market is seen as a passive counterparty accepting the opposite side of zero cost trades proposed by economic agents or market participants. Cash flows to trades are modeled as bounded random variables on a fixed probability space \((\Omega, \mathcal{F}, P)\) for a base probability measure that is in fact a risk neutral measure identified or selected by the economy.

The market is modeled by describing all the cash flows the market will accept at zero cost as a counterparty. Since market participants may trade in any size this set of cash flows is closed under scaling by any positive multiple. Hence it constitutes a cone of random variables. If the market accepts two cash flows \(X, Y\) it will accept the sum and hence this set of cash flows is a convex cone. Finally, the market will accept at zero cost any cash flow that is nonnegative and so the convex cone contains the nonnegative cash flows.

We therefore have a special structure for the zero cost cash flows acceptable to the market as a counterparty. This is that of a convex cone containing the nonnegative cash flows. The classical model for the market with its law of one price goes a step further and asserts that if a cash flow \(X\) is just acceptable to
the market with \( E^P[X] = 0 \) then as we trade in both directions at the same price, \( -X \) is also just acceptable and so the set of acceptable cash flows is identified with the half space defined by the condition \( E^P[X] \geq 0 \) or the set of all positive alpha trades as seen by the risk neutral measure \( P \) (Jensen (1968)). For two price markets we stop short of asserting the law of one price and hence the set of cash flows acceptable at zero cost is a proper convex cone containing the nonnegative cash flows. We denote by \( \mathcal{A} \) this set of cash flows acceptable to the market at zero cost. It will be smaller than the classical set of positive alpha trades and characteristically if \( X \) is just acceptable then \( -X \) will not be acceptable. One cannot reverse direction on the same terms.

A more constructive and equivalent characterization of such cash flows (i.e. those that are given by convex cones containing the nonnegative cash flows) was provided by Artzner, Delbaen, Eber and Heath (1999). It was shown there that for any such set of acceptable risks \( \mathcal{A} \), there exists a convex set \( \mathcal{M} \) of probability measures \( Q \in \mathcal{M} \), \( Q \) equivalent to \( P \), (that are referred to as supporting measures in Cherny and Madan (2010) or test measures in Carr, Geman, Madan (2001)) with the property that \( X \in \mathcal{A} \) if and only if

\[
E^Q[X] \geq 0, \text{ all } Q \in \mathcal{M}.
\]


One may now usefully contrast our model for the zero cost marketed cash flows with other models that have appeared in the finance literature. In one sense the complete markets model may be viewed as accepting just the identically zero cash flow. It would not take a negative and no one would give it a positive cash flow for free anyway. The focus is then reminiscent of market clearing, or hedging or perfect replication. A relaxation is made in equilibrium models as already noted to positive alpha trades constituting the set of zero cost cash flows acceptable to the market. At the other extreme we move to incomplete markets with their necessity of residual cash flows. In this context much has been written, for example, on superreplication with the associated view that only positive cash flows are acceptable to the market at zero cost (Broadie, Cvitanic, Soner (1998)). Now between positive alpha trades and the nonnegative cash flows lie our proposed models for the zero cost marketed cash flows.

For an arbitrary cash flow \( X \) Cherny and Madan (2010) show that in the absence of hedging assets the market bid \( b(X) \) and ask \( a(X) \) prices must satisfy

\[
b(X) = \inf_{Q \in \mathcal{M}} E^Q[X] \tag{2}
\]

\[
a(X) = \sup_{Q \in \mathcal{M}} E^Q[X]. \tag{3}
\]

These facts follow on noting that both \( a(X) - X \) and \( X - b(X) \) must belong to the set \( \mathcal{A} \), and furthermore by competition in the market one seeks the lowest possible ask and the largest possible bid prices.
In the presence of a collection $\mathcal{H}$ of zero cost hedging assets a special role is played by the risk neutral measures $\mathcal{R}$, defined as the set of all measures $Q$ equivalent to $P$ for which $E^Q[H] = 0$ for all $H \in \mathcal{H}$. In the presence of hedging assets we have that

$$b(X) = \inf_{Q \in \mathcal{M} \cap \mathcal{R}} E^Q[X]$$

$$a(X) = \sup_{Q \in \mathcal{M} \cap \mathcal{R}} E^Q[X].$$

In the presence of hedging assets

$$b(X) = \sup \{b : \text{there exists } H \in \mathcal{H} \text{ such that } X - b - H \in \mathcal{A} \}$$

$$a(X) = \inf \{a : \text{there exists } H \in \mathcal{H} \text{ such that } a + H - X \in \mathcal{A} \}.$$

The lower and upper hedges denoted $H$, $\overline{H}$ respectively satisfy

$$X - b(X) - H \in \mathcal{A}$$

$$a(X) + \overline{H} - X \in \mathcal{A}.$$

For an operational and explicit definition of a cone of acceptable risks one may turn to Kusuoka (2001) that characterizes all cones of acceptable risks where the acceptability of a random variable is defined solely by its probability law. This characterization is used by Cherny and Madan (2010) to define acceptability using concave distortions. Such a modeling perspective for markets ignores risk covariations with endowment or background risk positions (Franke, Stapleton and Subrahmanyam (1998), Basak (2000)). In this regard we note that the Walrasian auctioneer was not concerned with background risk either but just with avoiding shortfalls. Our model for the market similarly focuses on just the structure of cash flow probabilities with little concern for when they occur so long as they are small. More complex models working with state dependent cones of acceptable risks are possible and in particular one may seek to work with distributions conditional on the level of factors. Here we work in the first instance with the most basic model for the two price market.

Proceeding with a model based on distortions, we fix a concave distribution function $\Psi(u)$ defined on the unit interval that maps to the unit interval. A random variable $X$ with distribution function $F(x)$ is then defined to be acceptable just if

$$\int_{-\infty}^{\infty} x d\Psi(F(x)) \geq 0.$$  

(10)

The corresponding set of supporting or test measures $\mathcal{M}$ is defined (Cherny (2006)) by all change of measure densities $Z \geq 0$, $E^P[Z] = 1$ satisfying

$$E^P \left[ (Z - a)^+ \right] \leq \Phi(a) = \sup_{0 \leq u \leq 1} (\Psi(u) - ua), \text{ for all } a \geq 0.$$  

(11)

If attention is restricted to the class of acceptable distribution functions $F(x)$ or their inverses $G(u)$ then for the acceptability of $G$ one must have a positive
expectation under all change of measure densities on the unit interval $Z(u) \geq 0, \int_0^1 Z(u)du = 1$, for which $L' = Z$ satisfies $L(u) \leq \Psi(u)$ for $0 \leq u \leq 1$.

The bid and ask prices for $X$ with a hedging space $\mathcal{H}$ are now given in terms of the distortion by

$$b(X) = \sup_{H \in \mathcal{H}} \int_{-\infty}^{\infty} xd\Psi(F_{X-H}(x))$$

$$a(X) = \inf_{H \in \mathcal{H}} -\int_{-\infty}^{\infty} xd\Psi(F_{H-X}(x)).$$

In the absence of hedging assets one merely computes the relevant expectations with $H = 0$. The computation of the distorted expectation is facilitated in terms of an ordered sample from the relevant distribution function with $x_1 < x_2 < \cdots x_N$ as

$$\sum_{i=1}^N x_i \left( \Psi\left( \frac{i}{N} \right) - \Psi\left( \frac{i-1}{N} \right) \right).$$

### 3 Bid and Ask Prices in terms of the inverse distribution function

We now develop expressions for the bid and ask prices in terms of the inverse distribution function $G(u)$ of a hedged cash flow $X$ with distribution function $F(x)$. For this we define the median $m = G(.5)$, or $F(m) = .5$. We now write

$$b(X) = \int_{-\infty}^{\infty} xd\Psi(F(x))$$

$$= \int_{0}^{1} G(u)d\Psi(u)$$

$$a(X) = -\int_{-\infty}^{\infty} xd\Psi(1-F(-x))$$

$$= \int_{0}^{1} G(u)d\Psi(1-u)$$

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We now partition the integrals over the unit interval into integrals over \([0, 0.5]\) and \([0.5, 1]\) and then integrate by parts to get

\[
\begin{align*}
b(X) &= \int_0^{0.5} G(u) d\Psi(u) + \int_{0.5}^1 G(u) d\Psi(u) \\
&= G(u)\Psi(u)\bigg|_0^{0.5} - \int_0^{0.5} \Psi(u) dG(u) \\
&\quad + G(u)(\Psi(u) - 1)\bigg|_{0.5}^1 - \int_{0.5}^1 (\Psi(u) - 1) dG(u) \\
&= m + \int_0^1 \left(1_{u \geq 0.5} - \Psi(u)\right) dG(u) \quad \text{(17)}
\end{align*}
\]

Similarly we may write

\[
\begin{align*}
a(X) &= \int_0^{0.5} G(u) d\Psi(1-u) + \int_{0.5}^1 G(u) d\Psi(1-u) \\
&= -G(u)(\Psi(1-u) - 1)\bigg|_0^{0.5} + \int_0^{0.5} (\Psi(1-u) - 1) dG(u) \\
&\quad - G(u)\Psi(1-u)\bigg|_{0.5}^1 + \int_{0.5}^1 (\Psi(1-u) - 1) dG(u) \\
&= m + \int_0^1 \left(\Psi(1-u) - 1_{u \leq 0.5}\right) dG(u) \quad \text{(18)}
\end{align*}
\]

When a cash flow is increased by a constant, both the ask and bid prices rise by this constant as is clear from the general definitions of these prices provided by equations (4) and (5). In the expressions (17) and (18) this is captured by the move in the median. The rest of these expressions account for the charge related to the risk exposure. It is instructive in this regard to consider first a linear distribution function of the form

\[
F(x) = \frac{(x - a)^+ - (x - b)^+}{(b - a)}
\]

of a random variable uniformly distributed in the interval \((a, b)\). The inverse distribution function is just

\[
G(u) = a + (b - a)u
\]

with the median being

\[
m = \frac{a + b}{2}
\]

and the risk charge embedded in the ask and bid prices is just proportional to
the length of the interval of uncertainty with

\[ b(X) = \frac{a + b}{2} + (b - a) \int_0^1 \left( 1_{u \geq \frac{1}{2}} - \Psi(u) \right) du \]

\[ a(X) = \frac{a + b}{2} + (b - a) \int_0^1 \left( \Psi(1 - u) - 1_{u \leq \frac{1}{2}} \right) du \]

More generally for a piecewise linear distribution function taking the level \( p_i \) at the point \( a_i \) with \( p_0 = 0 \) and \( p_N = 1 \) we have that

\[ F(x) = \sum_{i=1}^{N} \left( p_{i-1} + (p_i - p_{i-1}) \frac{(x - a_{i-1})^+ - (x - a_i)^+}{(a_i - a_{i-1})} \right) \]

\[ G(u) = a_0 + \sum_{i=1}^{n} \left( \frac{a_i - a_{i-1}}{p_i - p_{i-1}} \left( (u - p_{i-1})^+ - (u - p_i)^+ \right) \right) \]

The ask and bid prices are now given by

\[ b(X) = m + \sum_{i=1}^{N} \frac{a_i - a_{i-1}}{p_i - p_{i-1}} \int_{p_{i-1}}^{p_i} \left( 1_{u \geq \frac{1}{2}} - \Psi(u) \right) du \]

\[ a(X) = m + \sum_{i=1}^{N} \frac{a_i - a_{i-1}}{p_i - p_{i-1}} \int_{p_{i-1}}^{p_i} \left( \Psi(1 - u) - 1_{u \leq \frac{1}{2}} \right) du \]

and we have an exposure to a sum of uniform variates with probabilities \( p_i - p_{i-1} \) over the interval \((a_{i-1}, a_i)\). The risk charge is then proportional to the interval length \((a_i - a_{i-1})\) and the charge is for the bid and ask respectively

\[ \frac{1}{p_i - p_{i-1}} \int_{p_{i-1}}^{p_i} \left( 1_{u \geq \frac{1}{2}} - \Psi(u) \right) du, \quad \frac{1}{p_i - p_{i-1}} \int_{p_{i-1}}^{p_i} \left( \Psi(1 - u) - 1_{u \leq \frac{1}{2}} \right) du. \]

In the limit the charge is

\[ \left( 1_{u \geq \frac{1}{2}} - \Psi(u) \right), \left( \Psi(1 - u) - 1_{u \leq \frac{1}{2}} \right) \]

for the interval \( dG(u) \).

The derivative of the inverse distribution function measures the sensitivity of the cash flow to a change in the quantile and is a measure of the risk exposure at the particular quantile. A zero derivative representing no risk exposure and no risk charge. The particular distortion determines the charge per unit exposure for each quantile.

### 4 Profits and Capital

We now define the level of cash reserves supporting trades and the amount of up-front profits that may be associated with positions in terms of bid and ask prices.
Consider first case of profits. We view a random variable defined as a state
contingent cash flow as produced at a cost given by the risk neutral expectation
or $E^P[X]$. Now this cash flow could be sold to the two price market at the market
bid price $b(X)$. We see the market as successfully selling it to counterparties at
the market ask price $a(X)$ and thereby earning the spread. The market however
is not a person and has no needs for funds and redistributes the spread among
the two parties of the trade. Hence we receive from the market the bid $b(X)$ the
counterparty pays the ask $a(X)$ but then the market distributes half the spread
to each of us with the net cost being the mid price and our receipts also being
equal to the mid price

$$m(X) = \frac{a(X) + b(X)}{2}.$$  \hfill (19)

The profit on the trade is then

$$\pi(X) = m(X) - E^P[X].$$  \hfill (20)

We now come to the definition of the level of supporting cash reserves. Again
if $X$ is produced and sold to market we expect to receive for it the bid price
$b(X)$. If the trade was to be unwound and we have to buy back $X$ from
the market we would have to pay for it $a(X)$. To guard against such an unfavorable
unwind we should hold reserves in the amount $a(X) - b(X)$. Therefore we define
the cash reserve capital to be

$$\kappa(X) = a(X) - b(X).$$  \hfill (21)

We now observe that an increase in a cash flow by a constant raises the mid
price and the risk neutral expectation by the same amount leaving the profit
unchanged. The same is true for the cash reserves as the ask and bid price rise
by the same amount. Hence both the profit and the capital are dependent solely
on the risk exposures embedded in the slope of the inverse distribution function.
It is useful to identify these functions.

Applying the same analysis to the expectation under $P$ as we did for the
distorted expectation in the last section we observe that

$$E^P[X] = m + \int_0^1 \left(1_{u > \frac{1}{2}} - u\right) dG(u).$$

We may now write

$$\pi(X) = \int_0^1 H(u)dG(u)$$

$$\kappa(X) = \int_0^1 K(u)dG(u)$$

where

$$H(u) = \left(\frac{\Psi(1-u) - 1_{u < \frac{1}{2}} + 1_{u \geq \frac{1}{2}} - \Psi(u)}{2} - \left(1_{u > \frac{1}{2}} - u\right)\right)$$

$$K(u) = \Psi(1-u) + \Psi(u) - 1$$

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The rate of return is then easily defined by

\[ \rho(X) = \frac{\pi(X)}{\kappa(X)}. \]

We observe that \( K(u) \) is symmetric about \( \frac{1}{2} \), with

\[ K(u) = K(1 - u), \]

and the function \( H(u) \) is antisymmetric with

\[ H(u) = -H(1 - u). \]

Furthermore we have that \( H(0) = H(1/2) = H(1) = 0 \). The function \( H \) is negative for \( u < 1/2 \) and positive for \( H > 1/2 \). We see that sensitivity of cash flows to quantiles above 0.5 are exposures to gain leading to profits while sensitivities to quantiles below 0.5 are loss exposures with a marked loss.

We now consider some sample distortions. Cherny and Madan (2009) introduced a family of distortions indexed by a parameter \( \gamma \) that defined a decreasing sequence of sets of acceptable risks \( \mathcal{A}_\gamma \), starting with the half space of positive expectation under \( P \) for \( \gamma = 0 \) and tending to arbitrage or the nonnegative cash flows as \( \gamma \) tends to infinity. Further the distortions were organized with an infinite derivative near zero and zero derivative near unity to incorporate a reweighting upwards to infinity for large losses and a reweighting downwards towards zero for large gains. Such a family of distortions incorporates both risk aversion in the market and an absence of gain enticement in the market. An example of such a distortion is \( \text{MINMAXVAR} \) for which

\[ \Psi^\gamma(u) = 1 - \left(1 - u^{\frac{1}{1+\gamma}}\right)^{1+\gamma}. \]

We may separate loss aversion \( \lambda \) and from absence of gain enticement \( \gamma \) and define

\[ \Psi^{\lambda,\gamma}(u) = 1 - \left(1 - u^{\frac{1}{1+\lambda}}\right)^{1+\gamma}. \]

We display in Figures 1 and 2 a sample of the functions \( H, K \) for the distortion \( \text{MINMAXVAR} \).

5 Capital and Leverage

We report in this section on the relationship between our capital assessments and the leverage being granted by such capital charges. For this purpose we develop first a measure of the scale of operations. In this regard we note that all our measures of profit, capital do not respond to constants. We consider a measure of scale that is equally unresponsive. Given that we finally work on the unit interval we use for the purpose of removing constants the deviation from
Figure 1: The profit charge on quantiles for MINMAXVAR at three stress levels of 0.1, 0.25 and 0.5
Figure 2: Capital charges for different quantile levels for MINMAXVAR at three stress levels of 0.1, 0.25 and 0.5.
the median \( m \). We take as our measure for scale the expectation of the deviation from the median or with density \( f(x) \) we have

\[
scale = \int_{-\infty}^{\infty} |x - m| f(x) dx.
\]

We now bring this computation back to the unit interval as follows.

\[
\int_{-\infty}^{\infty} |x - m| f(x) dx = \int_{-\infty}^{\infty} |y| f(m + y) dy \\
= -\int_{-\infty}^{0} y f(m + y) dy + \int_{0}^{\infty} y f(m + y) dy \\
= \int_{-\infty}^{0} F(m + y) dy + \int_{0}^{\infty} (1 - F(m + y)) dy \\
= \int_{0}^{1/2} udG(u) + \int_{1/2}^{1} (1 - u)dG(u).
\]

Define by

\[
S(u) = u \mathbf{1}_{u \leq \frac{1}{2}} + (1 - u) \mathbf{1}_{u \geq \frac{1}{2}}
\]
then we have that

\[
scale = \int_{0}^{1} S(u) dG(u).
\]

The leverage may be measured by the ratio of the scale to capital. This ratio is invariant to both shift and scale as both the numerator and denominator are invariant to shifts and they both scale and so the ratio is invariant. We may then measure the leverage with respect to a standard normal density for which \( G(u) \) is the inverse of the standard normal distribution.

We present in Figure 3 a graph of \( S(u) \) against \( K(u) \) for various settings of the stress parameter \( \gamma \). We observe that at \( \gamma = .5 \) one has stopped providing any leverage as the capital function \( K(u) \) completely dominates the scale function \( S(u) \). The domination occurs at \( \gamma = 0.4405 \). For lower values of \( \gamma \) some leverage is provided. For the values of \( \gamma \) at 0.025, 0.05, 0.1, 0.25, 0.4405 and 0.5 the leverage levels for a standard Gaussian are respectively 8.9305, 4.5045, 2.2908, 0.9605, 0.5751 and 0.5146 as obtained by integrating the kernels \( S, K \) against the inverse of the standard normal distribution function on the unit interval.

### 6 A Sample of Profit, Capital Leverage and Return Computations under geometric Brownian motion

We begin with a case of exposure to loss and a negative profit. Let us consider the debt claim under the geometric Brownian motion model of Black and Scholes (1973) and Merton (1973) with

\[
S = 100 \exp(.2Z - .04/2)
\]
where $Z$ is a standard normal variate and the cash flow given by

$$X = \min(S, 80).$$

For this cash flow the client takes loss and we take gain so we discount and the values are for minmaxvar at 0.75 as follows.

$$b(X) = 77.4751$$  
$$a(X) = 79.5226$$  
$$m(X) = 78.4989$$  
$$E^p[X] = 78.8100$$  
$$\pi(X) = -0.3111$$  
$$\kappa(X) = 2.0475$$  
$$\lambda(X) = 0.5812$$  
$$\rho(X) = -1.519.$$  

for the bid, ask, mid, expectation, profit, capital and return respectively.

We present the rate of return for out of the money options under geometric Brownian motion with a 20% volatility for a variety of strikes and maturities in Figure 4. The computations are for the distortion minmaxvar at the stress level of 0.25. We have 41 strikes ranging from 80 to 120 in steps of a dollar and 36 maturities from a quarter to 2 years in steps of .05. These are all assets with a gain exposure with a corresponding positive profit level and rate of return.
Figure 4: Graph of rates of return on a GBM risk neutral stock with a 20% volatility for a variety of strikes and maturities.
Figure 5 presents a graph of the profits on the same set of options under the same measure.

Figure 6 presents the capital required for the same options under the same measure.

Figure 7 presents the leverage granted for the same options under the same measure.

7 Variance Swap under the Variance Gamma Model

For an example of a contract that cannot be hedged but for which there exist good known hedges we consider the variance swap contract when the underlying process is the variance gamma Lévy process (Madan and Seneta (1990), Madan, Carr and Chang (1998)). Consider a daily variance swap on a stock driven by a Lévy process hedged using a position in the stock rebalanced daily. The risk
Figure 6: Capital required on out-of-the-money options for a variety of strikes and maturities under geometric Brownian motion at a 20% volatility.
Figure 7: Leverage granted on out-of-the-money options for a variety of strikes and maturities under geometric Brownian motion at a 20% volatility.
neutral law for the stock is given by

\[ S(t) = S(0) \exp(X(t) + \omega t) \]

where \( \omega \) is the convexity correction term. The process \( X(t) \) is given by

\[ X(t) = \theta g(t) + \sigma W(g(t)) \]

\( g(t) \) is the gamma process with unit mean rate and variance rate \( \nu \).

\[ \omega = \frac{1}{\nu} \ln \left( 1 - \theta \nu - \sigma^2 \nu / 2 \right) \]

The unhedged cash flow is

\[ \frac{252}{N} \sum_{i=1}^{N} \ln \left( \frac{S(t + ih)}{S(t + (i-1)h)} \right)^2 \]

for \( h \) given by a day. The hedged cash flow also subtracts

\[-2 \log(S(T)/S(0)) + 2E \log(S(T)/S(0)) + \sum_{i=1}^{M} \frac{2}{S(t + (i-1)h)} (S(t + ih) - S(t + (i-1)h)) \]

We present the results for the standard hedge (Neuberger (1990), Dupire (1992), Carr and Madan (1998), Derman, Demeterfi, Kamal and Zou (1999)) and an improved scaled hedge in two subsections.

### 7.1 The standard hedge

We compute all the entities of interest on both these cash flows. The computations were done for the \( VG \) process driving the stock with \( \sigma = .2, \nu = .75, \theta = -3 \). We computed all the entities for 8 maturities ranging from .25 to 2 years in steps of .25. The stress level used was 0.25 and the distortion was \text{minmaxvar}. We report the unhedged and hedged results in the following tables with prices reported as volatilities.

We observe from the table that the hedge allows one to raise the bid and lower the ask on all the contracts. The profit is reduced in the hedge but the capital required also falls raising rates of return on the earlier maturities. However, profit and the rate of return is negative on the longer maturities that have greater exposure to possible unhedged losses. Leverage rises at the longer maturities.

To verify the nature and quality of the hedge we present in Figure 8 a graph of the post hedge residual cash flow distribution function at the one and two year maturities. We observe that at one year we are hedged with some gain exposure while at two years we have a hedge with loss exposure. We graph for our purposes here the inverse of the distribution function.

We observe from the shape of these functions the good hedging region and the regions where we over hedge and under hedge. The former yields the mark of a profit while the latter yields a marked loss.
Unhedged Variance Swap with VG underlier $\sigma_g = .2$, $\nu = .75$, $\theta = -.3$

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<th>capital</th>
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Figure 8: Graph of the realized inverse cumulative distribution function for a variance swap with a VG underlier for a one and two year maturity.
7.2 Improving the hedge

Instead of hedging the variance swap with a factor of 2 we considered altering this factor for the two year position. We chose the scale factor with a view to minimizing first the ask price and then the capital committed. The resulting optimal positions were a scale of 1.2856 and 1.4732 in place of 2. For these positions the results for the entities of interest are respectively

\[
\begin{align*}
\text{bid} & = 0.3253, 0.3179 \\
\text{ask} & = 0.3434, 0.3471 \\
\text{mid} & = 0.3344, 0.3328 \\
\text{rne} & = 0.3329, 0.3336 \\
\text{prf} & = 11, -6 \text{ bp} \\
\text{cap} & = 121, 194 \text{ bp} \\
\text{lev} & = 0.6907, 0.9148 \\
\text{ror} & = 8.80\%, -2.84\%
\end{align*}
\]

For longer dated contracts driven by Lévy processes it is recommended that one lower the scale factor in the hedge.

The inverse distribution functions after such a scale adjustments are presented in Figure 9.

8 Hedging a Call option under the Variance Gamma process

We take a one year 110 call on this path space and hedge each of the four quarters using stock positions that are interpolated at each quarter from the following points for the spot and the delta. The optimization criterion was the minimization of the post hedge ask price using \textit{minmaxvar} at the stress level 0.5.

The initial delta was 0.2712. For the second quarter we have

\[
\begin{align*}
\text{stock} & \quad 51.56 \quad 104.83 \quad 120.00 \\
\text{delta} & \quad 0 \quad 0.2858 \quad 0.4445
\end{align*}
\]

The third quarter positions are interpolated from

\[
\begin{align*}
\text{stock} & \quad 41.69 \quad 105.63 \quad 131.10 \\
\text{delta} & \quad 0 \quad 0.1796 \quad 0.7270
\end{align*}
\]

The last quarter is given by

\[
\begin{align*}
\text{stock} & \quad 35.85 \quad 105.06 \quad 140.87 \\
\text{delta} & \quad 0 \quad 0.1063 \quad 0.8422
\end{align*}
\]
Figure 9: Graph of inverse CDF on variance swap with VG underlying for 2 year maturity after adjustment of hedge scale down from 2.0 to 1.31.
In each case we extrapolated delta linearly for prices outside the interpolation interval and then floored the delta at zero and capped it at unity.

The unhedged and hedged entities of interest are presented below. Also presented are the results for minimizing capital commitments and maximizing expected utility for a unit risk aversion coefficient. The graphs presents all four inverse distribution functions. For the minimum capital hedge and the maximization of expected utility the delta hedging was monthly in place of quarterly. Given the interest in minimizing the ask price and maximizing the bid price one may as well consider minimizing the sum of the ask price and the negative of the bid price which is the capital required. From an economic perspective economizing capital is a good perspective for a market objective function in contrast to expected utility maximization. We see in this example that this particular expected utility maximization criterion over commits capital by some 18.89%. In any case if one wishes to economize capital then this should be the direct objective function for the design of hedges.

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<th>Hedged Min Cap</th>
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The inverse cdfs are presented in Figure 10.

For the minimum capital hedge and the maximum expected utility hedge we traded the underlying monthly for the twelve months. The initial delta for the minimum capital hedge was 0.2266, while for maximum expected utility it was 0.1661. The other deltas are given by two matrices for the spot levels and the corresponding delta levels.

9 Delta Hedging in Left Skewed Markets

In this section we analyze the effect on delta hedging policies of the existence of left skewed risk neutral distributions. We consider hedging risk exposures over a week or five days with a position in the underlying stock. The cash flows to be hedged have quadratic value functions defined by their deltas and gammas. We take 21 settings for the delta from $-1$ to $1$ in steps of 0.1. There are 41 settings for the gamma ranging from negative 5 to 5 in steps of .25. This gives us a set of 861 cash flows to hedge. The stock starts at 100 and the target cash flow 5 days out is

$$c(S) = \delta(S - 100) + \frac{\gamma}{2}(S - 100)^2.$$

We generate readings for the stock price five days later from the variance gamma process for the stock with parameter setting $\sigma = 0.2$, $\nu = 0.75$ and two
Figure 10: Graph of the inverse cdf for an unhedged and hedged call for an underlying VG process. The red curve is for a quarterly hedge minimizing the ask price while the blue curve is a monthly hedge minimizing the capital required. The green curve is from maximizing exponential expected utility with unit risk aversion.
### Call Spot Levels

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### Call Delta Levels

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## Call Delta Levels on Maximizing Expected Utility

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<th>0.1937</th>
<th>0.2323</th>
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Levels of skew $\theta = -0.1$, and $\theta = -0.3$. We generate 20000 readings from the variance gamma process five days out and eliminate outliers by only taking values for the stock price between 80 and 120. On this set of stock price possibilities we define a stock position of $a$ shares and the residual cash flow

$$r(S) = c(S) - a(S - 100).$$

We then compute the capital required to hold this residual cash flow and find the value of the optimal stock position that minimizes required capital. The starting position in the stock is the delta for the stock and we penalize changing this position by requiring an improvement in capital required for any change to the hedge position. The specific objective function minimized for the stock position is

$$\zeta(a) = a(r(S)) - b(r(S)) + 0.2 \times |a - \delta|.$$  

We thus obtain 861 optimal stock positions $a^*$ for each choice of the two choices for $\theta$, the skewness parameter of the volatility surface. We regress the optimal positions on the delta and the gamma of the cash flow being hedged to get the following results.

$$a^*_i = \delta_i - 0.8194\gamma_i \text{ for } \theta = -0.1$$

$$a^*_i = \delta_i - 1.9702\gamma_i \text{ for } \theta = -0.3.$$  

We thereby observe that in left skewed markets stock positions should be adjusted downwards in response to the gamma of cash flow to be hedged to minimize capital commitments.

## 10 Conclusion

The theory of two price markets as recently developed in Cherny and Madan (2010) provides closed forms for bid and ask prices for a state contingent cash
flow based on hedged residual cash flows being acceptable to markets. The concept of acceptability used was defined in Artzner, Delbaen, Eber and Heath (1999) and we employ an operational form based on concave distortions introduced in Cherny and Madan (2009). It is argued that when markets are viewed as passive impersonal counterparties sharing the spreads they earn with their trading counterparties then the profit on a trade may be seen as difference between the mid quote and the risk neutral expectation. This theoretical perspective permits the definition of up-front profits on trades. Furthermore, the cash reserve needed to unwind a sale to market at bid is seen to be the difference between the ask and the bid prices. We therefore also have a foundation for capital reserves and hence both leverage and rates of return on trades.

New expressions are developed for the bid and ask prices in terms of the sensitivity of the inverse distribution function to the quantile level. This sensitivity turns out to be a measure of risk exposure at the quantile level and both profits and capital are quantile based charges integrated over the quantiles for this risk exposure. The profit charge is positive for gain quantiles above the median and is negative for loss quantiles below the median. Capital charges are positive at all quantiles but fall to zero in the extremes and are highest near the median.

The theory is illustrated on unhedged exposures in the Black Merton Scholes model, followed by variance swaps, call options for variance gamma underliers and capital minimizing revisions for delta hedging in left skewed markets. The competitive pressures to minimize ask prices and maximize bid translate into market objective functions to economize capital. It is observed that the maximization of expected utility as a proxy criterion for the market may result in uneconomic capital levels and hedges should be designed to economize capital. The theory thus offers new objectives for the design of hedges than has heretofore been possible. We note that stock positions should be revised downwards from zero delta in left skewed markets in response to the target gamma with a view to minimizing capital.

The presentation here is in a static context and dynamic generalizations are possible. We expect they will involve the theory of nonlinear expectations.

References


