Rate-Memory Trade-off for the Two-User Broadcast Caching Network with Correlated Sources

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Abstract—This paper studies the fundamental limits of caching in a network with two receivers and two files generated by a two-component discrete memoryless source with arbitrary joint distribution. Each receiver is equipped with a cache of equal capacity, and the requested files are delivered over a shared error-free broadcast link. First, a lower bound on the optimal rate-memory trade-off is provided. Then, in order to leverage the correlation among the library files to alleviate the load over the shared link, a two-step cache-aided coded multicast (CACM) scheme is proposed. The first step uses Gray-Wyner source coding to represent the library via one common and two private descriptions, such that a second correlation-unaware CACM step can exploit the additional coded multicast opportunities that arise. It is shown that the rate achieved by the proposed two-step scheme matches the lower bound for a significant memory regime and it is within half of the conditional entropy for all other memory values.

I. INTRODUCTION AND SETUP

The use of caching at the wireless network edge has emerged as a promising approach to efficiently increase the capacity of wireless access networks. There have been extensive studies characterizing the fundamental rate-memory trade-off in a broadcast caching network with a library composed of independent content [1]–[4]. More recently, [5]–[7] have studied the rate-memory trade-off when delivering correlated files. In [5], the authors consider a single-receiver multiple-file network with lossy reconstructions and characterize the trade-offs between rate, cache capacity, and reconstruction distortions. They extend the analysis to a scenario with two receivers and one cache, in which local caching gains can be exploited. The works in [6] and [7] consider a setting with an arbitrary number of files and receivers each having a cache. A practical correlation-aware scheme is proposed in [6], in which content is cached according to both the popularity of files and their correlation with the rest of the library. Then, the requested content is delivered via a multicast codeword composed of compressed versions of the requested files. Alternatively, the work in [7] addresses the dependency among content files by first compressing the correlated library, which is then treated as a library of independent files by a conventional cache-aided coded multicast scheme.

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In this paper, by focusing on a two-user two-file setting, we are able to characterize the optimal worst-case rate-memory trade-off of the broadcast caching network with correlated sources. To this end, we first provide a lower bound on the optimal worst case rate-memory trade-off, which is derived using a cut-set argument on the corresponding cache-demand augmented graph [8]. The lower bound improves the best known bound for correlated sources given in [9], and when particularized to independent sources, matches the corresponding best known bound derived in [10]. We then propose a two-step scheme, in which the source files are first encoded based on the Gray-Wyner network [11], and in the second step, they are cached and delivered through a multiple-request correlation-unaware cache-aided coded multicast scheme. In the rest of the paper we discuss the optimality of the proposed two-step scheme by characterizing a lower bound on the rate-memory trade-off for this class of schemes, and comparing it with the lower bound on the optimal trade-off. We identify the set of operating points in the achievable Gray-Wyner region [11], [12], for which a two-step scheme is optimal over a range of cache capacities, and approximates the optimal rate to within half of the conditional entropy for all cache sizes.

The paper is organized as follows. Sec. II presents the information-theoretic problem formulation. In Sec. III, we introduce a class of two-step schemes based on the Gray-Wyner network. The lower bounds for the optimal and two-step schemes are provided in Sec. IV, and later used to establish the optimality of an achievable two-step scheme proposed in Sec. V. After analyzing an illustrative example in Sec. V-A, the paper is concluded in Sec. VI.

II. NETWORK MODEL AND PROBLEM FORMULATION

We consider a broadcast caching network composed of one sender with access to a library of two files generated by a two-component discrete memoryless source (2-DMS). The 2-DMS model \( (X_1 \times X_2, p(x_1, x_2)) \) consists of two finite alphabets \( X_1, X_2 \) and a joint pmf \( p(x_1, x_2) \) over \( X_1 \times X_2 \). The 2-DMS generates i.i.d. random processes \( \{X_{1i}, X_{2i}\} \) with \( (X_1, X_2) \sim p(x_1, x_2) \). The two library files are represented by sequences \( X^n_1 = (X_{11}, \ldots, X_{1n}) \) and \( X^n_2 = (X_{21}, \ldots, X_{2n}) \), respectively, where \( X_{1i} \in X^n_1 \) and \( X_{2i} \in X^n_2 \). The sender communicates with two receivers \( r_1 \) and \( r_2 \) over a shared error-free broadcast link. Each receiver is equipped with a cache of size \( nM \) bits, where \( M \) denotes the (normalized) cache capacity.

We assume that the system operates in two phases: a caching phase and a delivery phase. During the caching phase, which
takes place at off-peak hours when network resource are abundant, the receiver caches are filled with functions of the library files, such that during the delivery phase, when user demands are revealed and resources are limited, the sender broadcasts the shortest possible codeword that allows each receiver to losslessly recover its requested file. We refer to the overall scheme as a cache-aided coded multicast scheme (CACM). Given a realization of the library, \(\{X_1^n, X_2^n\}\), a CACM scheme consists of the following components:

- **Cache Encoder**: During the caching phase, the cache encoder designs the cache content of receiver \(r_i\) using a mapping \(f_r^n : X_1^n \times X_2^n \rightarrow [1 : 2^{nR}]\). The cache configuration of receiver \(r_i\) is denoted by \(Z_{r_i} = f_r^n(X_1^n, X_2^n)\).

- **Multicast Encoder**: During the delivery phase, each receiver requests a file from the library. The demand realization, denoted by \(d = (d_{r_1}, d_{r_2}) \in D = \{1, 2\}^2\), where \(d_{r_i} \in \{1, 2\}\) denotes the index of the file requested by receiver \(r_i\), is revealed to the sender, which then uses a fixed-to-variable mapping \(f_d^n : D \times X_1^n \times X_2^n \rightarrow [1 : 2^{nR}]\) to generate and transmit a multicast codeword \(W_d = f_d^n(d, \{Z_{r_1}, Z_{r_2}\}, \{X_1^n, X_2^n\})\) over the shared link. The codeword \(W_d\) is designed for each demand realization according to the cache content, library files, and joint distribution \(p(x_1, x_2)\), to enable almost-lossless reconstruction of the requested files.

- **Multicast Decoders**: Each receiver \(r_i\) uses a mapping \(g_{d_{r_i}}^n : D \times X^n \rightarrow [1 : 2^{nR}]\) to recover its requested file, \(X_{d_{r_i}}^n\), using the received multicast codeword and its cache content as \(X_{d_{r_i}}^n = g_{d_{r_i}}^n(d, W_d, Z_{d_{r_i}})\).

The worst-case probability of error of a CACM scheme for demand \(d\) is given by

\[
P_e(n) = \max_d \max_{r_i} \Pr\left(\hat{X}_{d_{r_i}}^n \neq X_{d_{r_i}}^n\right).
\]

In this paper, we focus on the worst-case multicast rate,

\[
R(n) = \max_d \frac{\mathbb{E}[|W_d|]}{n},
\]

where \(L(W)\) denotes the length (in bits) of the multicast codeword \(W\), and the expectation is over the library files.

**Definition 1**: A worst-case rate-memory pair \((R, M)\) is achievable if there exists a sequence of CACM schemes for cache capacity \(M\) and increasing file size \(n\), such that \(\lim_{n \to \infty} R(n) = 0\), and \(\limsup_{n \to \infty} R(n) \leq R\).

**Definition 2**: The worst-case rate-memory region, \(R^*\), is the closure of the set of achievable worst-case rate-memory pairs \((R, M)\), and the optimal worst-case rate-memory function is

\[
R^*(M) = \inf\{R : (R, M) \in R^*\}.
\]

### III. Gray-Wyner CACM Scheme

In this section, we describe a class of CACM schemes based on a two-step lossless source coding setup, as depicted in Fig. 1. The first step involves classical Gray-Wyner source coding [12], and the second step, lossless (correlation-unaware) source coding with distributed side information [1], [3], [9]. We refer to this scheme as Gray-Wyner Cache-Aided Coded Multicast (GW-CACM).

The Gray-Wyner system depicted in Fig. 1 is a distributed lossless source coding setup in which a 2-DMSS \((X_1, X_2)\) is represented by three descriptions \(\{Y_0^{nR_0}, Y_1^{nR_1}, Y_2^{nR_2}\}\) such that file \(d_{r_i}\) can be losslessly recovered from descriptions \((Y_0^{nR_0}, Y_{nR_{d_{r_i}}})\), and file \(d_{r_2}\) can be losslessly recovered from descriptions \((Y_0^{nR_0}, Y_{nR_{d_{r_2}}})\), both asymptotically, as \(n \to \infty\).

As shown in [12], the Gray-Wyner rate region is the closure of the union over \(U\) of \(\mathcal{S}_{GW}(U)\), where \(\mathcal{S}_{GW}(U)\) denotes the set of rate triplets \((R_0, R_1, R_2)\) such that

\[
\begin{align*}
R_0 &\geq I(X_1, X_2; U), \\
R_1 &\geq H(X_1 | U), \\
R_2 &\geq H(X_2 | U),
\end{align*}
\]

given a conditional pmf \(p(u|x_1, x_2)\) with \(|U| \leq |X_1| + |X_2| + 2\).

For a given \(U\) and a rate triplet \((R_0, R_1, R_2) \in \mathcal{S}_{GW}(U)\), a GW-CACM scheme consists of:

- **Gray-Wyner Encoder**: Given a library realization \(\{X_1^n, X_2^n\}\), the Gray-Wyner encoder at the sender computes three descriptions \(\{Y_0^{nR_0}, Y_1^{nR_1}, Y_2^{nR_2}\}\) using a mapping \(f_{GW} : X_1^n \times X_2^n \rightarrow [1 : 2^{nR_0}] \times [1 : 2^{nR_1}] \times [1 : 2^{nR_2}]\).

- **Cache Encoder**: Given the descriptions \(\{Y_0^{nR_0}, Y_1^{nR_1}, Y_2^{nR_2}\}\), the cache encoder at the sender computes the Gray-Wyner based cache contents \(Z_{r_i} = f_{GW}^{-1}(X_1^n, X_2^n)\) where \(r_i \in \{1, 2\}\).

- **Multicast Encoder**: For any demand realization \(d\) revealed to the sender, the Gray-Wyner based multicast encoder generates and transmits the multicast codeword \(W_d^{GW} = f_{GW}(d, \{Z_{r_1}, Z_{r_2}\}, \{Y_0^{nR_0}, Y_1^{nR_1}, Y_2^{nR_2}\})\).

- **Multicast Decoder**: Receiver \(r_i\) decodes the descriptions corresponding to its requested file as

\[
\hat{Y}_{nR_0} = g_{d_{r_i}}^{GW}(d, W_d^{GW}, Z_{r_i}).
\]

- **Gray-Wyner Decoder**: Receiver \(r_i\) decodes its requested file using the descriptions recovered by the multicast decoder, via a mapping \(g_{d_{r_i}}^{GW} : [1 : 2^{nR_0}] \times [1 : 2^{nR_{d_{r_i}}}] \rightarrow X_1^n\), as

\[
\hat{X}_{d_{r_i}} = g_{d_{r_i}}^{GW}(Y_0^{nR_0}, Y_{d_{r_i}}).
\]
Notice that for the class of GW-CACM schemes, since \((R_0, R_1, R_2) \in \mathcal{G}_W(U)\) and \((Y_{0}^{nR_0}, Y_{d_1}^{nR_1})\) is a Gray-Wyner description of \(X_d\), in order to have \(\lim_{n \to \infty} P_e(n) = 0\), with \(P_e(n)\) as defined in (1), we only need
\[
\lim_{n \to \infty} \max_{d \in \mathcal{D}} \max_{r_i} \mathbb{P}\left(\tilde{Y}_{0}^{nR_0}, \tilde{Y}_{d_1}^{nR_1} \neq Y_{0}^{nR_0}, Y_{d_1}^{nR_1}\right) = 0.
\]
As in (2), the worst-case GW-CACM multicast is
\[
R_{GW}^*(R_0, R_1, R_2) = \max_{d \in \mathcal{D}} \frac{E[L(W_d^{GW})]}{n}, \tag{6}
\]
where we explicitly show the dependence on \((R_0, R_1, R_2)\).

In line with Definitions 1 and 2, for a given \(U\), the worst-case U-rate-memory region for the class of GW-CACM schemes, \(\mathcal{R}_{GW}(U)\), is defined as the closure of the union of all the achievable pairs \((R_{GW}(R_0, R_1, R_2), M)\) with \((R_0, R_1, R_2) \in \mathcal{G}_W(U)\). Analogously, the worst-case U-rate-memory function of GW-CACM, \(R_{GW}^*(M, U)\), is defined as \(R_{GW}^*(M, U) = \inf\{R : (R, M) \in \mathcal{R}_{GW}(U)\}\).

We remark that \(R_{GW}(M, U)\) is the rate achieved by a GW-CACM scheme with the Gray-Wyner encoder operating at the boundary of the region \(\mathcal{G}_W(U)\). Finally, optimizing over the choice of \(U\), we obtain the worst-case rate-memory region, \(\mathcal{R}_{GW}^*\), and the worst-case rate-memory function, \(R_{GW}^*(M)\), of GW-CACM as
\[
\mathcal{R}_{GW}^* = cl\left\{\bigcup \mathcal{R}_{GW}(U)\right\}, \quad R_{GW}^*(M) = \inf \{R_{GW}(M, U)\},
\]
where \(cl\{S\}\) denotes the closure of \(S\), and the union and infimum are over all choices of \(U\) with \(\min |U| \leq |X_1|, |X_2| + 2\).

### IV. LOWER BOUNDS

In this section, we provide lower bounds for \(R^*(M)\), the optimal worst-case rate-memory function, and \(R_{GW}^*(M, U)\), the worst-case U-rate-memory function of GW-CACM for a given \(U\). The latter bound can also be used to obtain a lower bound for \(R_{GW}^*(M)\). We then investigate conditions on the cache capacity \(M\) under which the lower bounds for \(R^*(M)\) and \(R_{GW}^*(M)\) meet. These conditions are then used in Section V in order to establish the optimality of GW-CACM, and quantify the rate gap from the lower bound as a function of the cache capacity \(M\).

#### A. Lower bound on \(R^*(M)\)

**Theorem 1:** For a broadcast caching network with two receivers, cache capacity \(M\), and a library composed of two files with joint distribution \(p(x_1, x_2)\), a lower bound on the optimal worst-case rate-memory function \(R^*(M)\) is given by
\[
R^{LB}(M) = \inf \left\{R : \begin{align*}
R &\geq H(X_1, X_2) - 2M, \\
R &\geq \frac{1}{2} \left(H(X_1, X_2) - M\right), \\
R &\geq \frac{1}{2} \left(H(X_1, X_2) + \max \left\{H(X_1), H(X_2)\right\}\right) - M.
\end{align*}\right\}
\]

**Proof 1:** Theorem 1 follows from combining i) cut-set bound arguments on the cache-demand-augmented graph described in [8], and ii) information-theoretic bounds obtained using cut-set arguments on the time-replication of the cache-demand-augmented graph as described in [13].

**Remark 1:** The outer bound in Theorem 1 improves the best known bound for correlated sources given in [9, Theorem 2], and when particularized to independent sources, matches the corresponding best known bound derived in [10].

#### B. Lower bounds on \(R_{GW}^*(M, U)\) and \(R_{GW}^*(M)\)

**Theorem 2:** For a given \(U\), a lower bound on the worst-case U-rate-memory function of the GW-CACM scheme, \(R_{GW}^*(M, U)\), is given by
\[
R^{LB}(M, U) = \inf \left\{R : \begin{align*}
R &\geq I(X_1, X_2; U) + H(X_1|U) + H(X_2|U) - 2M, \\
R &\geq \frac{1}{2} \left(I(X_1, X_2; U) + H(X_1|U) + H(X_2|U) - M\right), \\
R &\geq I(X_1, X_2; U) + H(X_1|U) + \frac{1}{2} H(X_2|U) - M, \\
R &\geq I(X_1, X_2; U) + \frac{1}{2} H(X_1|U) + H(X_2|U) - M.
\end{align*}\right\}
\]

**Proof 2:** The proof is similar to that of Theorem 1; now applied to Gray-Wyner descriptions at rates \((R_0, R_1, R_2) \in \mathcal{G}_W(U)\).

**Corollary 1:** A lower bound on \(R_{GW}^*(M)\), the worst-case rate-memory function of GW-CACM, is given by
\[
R^{LB}(M) = \inf \{R^{LB}(M, U)\},
\]
where the infimum is over all choices of \(U\) with \(\min |U| \leq |X_1|, |X_2| + 2\).

#### C. Where \(R^{LB}(M)\) and \(R^{LB}(M)\) meet

By comparing the lower bounds in Theorems 1 and 2, it is easy to see that \(R^{LB}(M) \leq R^{LB}(M, U)\), and hence, \(R^{LB}(M) \leq R^{LB}(M)\). In the following, we derive conditions under which \(R^{LB}(M) = R^{LB}(M)\).

**Theorem 3:** Let
\[
M_1 = \max_{X_1 \in U} \frac{1}{2} \min \left\{H(X_1|U), H(X_2|U)\right\},
\]
Then, for \(M \in [0, M_1]\) \([H(X_1, X_2) - 2M_1, H(X_1, X_2)]\), we have \(R^{LB}(M) = R^{LB}(M)\).

**Proof 3:** Theorem 3 follows from comparing \(R^{LB}(M)\) with \(R^{LB}(M, U)\) for a given \(U\), over different regions of memory \(M\). It is observed that when
\[
M \in [0, 1] \left[\min \left\{H(X_1|U), H(X_2|U)\right\}\right] \cup \left[H(X_1, X_2) - \min \left\{H(X_1|U), H(X_2|U)\right\}, H(X_1, X_2)\right],
\]
\(R^{LB}(M, U) - R^{LB}(M)\) is independent from the cache capacity \(M\), and becomes zero when \(I(X_1, X_2; U) + H(X_1|U) + H(X_2|U) = H(X_1, X_2)\). For the choice of \(U\) used to obtain \(M_1\), the region of memory over which the two bounds meet is maximized.
Remark 2: The Markov chain $X_1 - U - X_2$ in Theorem 3 suggests that for the rate triplet $(R_0, R_1, R_2) \in \mathcal{S}_{GW}(U)$ used in GW-CACM, we require $R_0 + R_1 + R_2 = H(X_1, X_2)$. The same Markov chain is also used to define Wyner’s common information [12]. While in Wyner’s common information the goal is to minimize $R_0$ subject to $R_0 + R_1 + R_2 = H(X_1, X_2)$, for $M_1$ in Theorem 3, the goal is to maximize $\min(R_1, R_2)$.

Theorem 4: If the file library $(X_2, X_3)$ is such that
\[(I(X_1;X_2), H(X_2|X_1), H(X_3|X_1)) \in cl \left\{ \bigcup \mathcal{S}_{GW}(U) \right\},\]
where the union is over all choices of $U$ with $|U| \leq |X_1|+|X_2|+2$, then $R_{LB}^U(M) = R_{GW}^U(M)$.

Proof 4: See [14].

Example 1: Consider a 2-DMS whose joint pmf $p(x_1, x_2)$ is such that $X_1, X_2$ can be represented as $X_1 = (X_1', V)$ and $X_2 = (X_2', V)$, where $X_1'$ and $X_2'$ are conditionally independent given $V$. Taking $U = V$, the point $R_0 = I(X_1, X_2; U) = H(V) = I(X_1; X_2)$, $R_1 = H(X_1|U) = H(X_1'|V) = H(X_1|X_2)$ and $R_2 = H(X_2|U) = H(X_2'|V) = H(X_2|X_1)$ belongs to the Gray-Wyner rate region. Then, by Theorem 4, $R_{LB}^U(M) = R_{GW}^U(M)$ for any $M$.

V. AN ACHIEVABLE SCHEME BASED ON GW-CACM AND ITS OPTIMALITY

In this section, we present an achievable GW-CACM scheme, where i) the first step consists of a Gray-Wyner encoder restricted to operate on the plane of the Gray-Wyner rate region with $R_1 = R_2 = \rho$, and ii) the second step is a deterministic correlation-unaware CACM scheme that combines LFU\(^2\) caching and unencoded (naive) multicasting with the best known correlation-unaware CACM scheme. We remark that while improved correlation-unaware schemes can be found in [3], [4], [9], [15], under the setting considered in this paper, the best known rate for integer cache size can be achieved by the deterministic CACM scheme of Maddah-Ali and Niesen [1], referred to as MN in the following. We then refer to the overall scheme as GW-LFU-MN, which works as follows:

Gray-Wyner Encoder: generates three library descriptions $\{Y_0^n, Y_1^n, Y_2^n\}$ using a conditional pmf $p(u|x_1, x_2)$ such that $p(x_1|u) = p(x_2|u)$ with $(R_0, \rho, \rho) \in \mathcal{S}_{GW}(U)$.

Cache Encoder: populates the user caches as:

- If $M \in [0, \rho]$, the common description $Y_0^n$ is not cached at either receiver, and descriptions $\{Y_1^n, Y_2^n\}$ are cached according to the caching phase of MN.
- If $M \in [\rho, R_0 + \rho)$, the first $(M - \rho)$ bits of description $Y_0^n$ are cached at both receivers (as per LFU caching), and descriptions $\{Y_1^n, Y_2^n\}$ are cached according to MN caching over the remaining cache capacity $\rho$.
- If $M \in [R_0 + \rho, R_0 + 2\rho]$, the common description $Y_0^n$ is fully cached at both receivers, and descriptions $\{Y_1^n, Y_2^n\}$ are cached according to MN caching over the remaining cache capacity $M - R_0$.

Multicast Encoder: transmits the descriptions $\{Y_1^n, Y_2^n\}$ according to the delivery phase of MN, while the portion of $Y_0^n$ missing at each receiver cache is transmitted via uncoded (naive) multicast.

Theorem 5: Given a conditional pmf $p(u|x_1, x_2)$ such that $p(x_1|u) = p(x_2|u)$, a cache capacity $M$, and a rate triplet $(R_0, \rho, \rho) \in \mathcal{S}_{GW}(U)$, the worst-case U-rate achieved by GW-LFU-MN is given by
\[R_{GW}^U(M, U) = \inf R_{ach}(R_0, \rho),\]
where the infimum is over all rate triplets $(R_0, \rho, \rho) \in \mathcal{S}_{GW}(U)$, and $R_{ach}(R_0, \rho)$ is
\[R_{ach}(R_0, \rho) = \begin{cases} R_0 + 2\rho - \frac{3}{2}M, & M \in [0, \rho) \\ R_0 + \frac{3}{2}\rho - M, & M \in [\rho, R_0 + \rho) \\ \frac{1}{2}R_0 + R_0 + \frac{1}{2}M, & M \in [R_0 + \rho, R_0 + 2\rho]. \end{cases}\]

Proof 5: The proof is given in [14], where it is shown that the cache allocation described above is optimal for the GW-LFU-MN scheme.

As in Secs. II and III, the worst-case rate achieved by GW-LFU-MN is given by
\[R_{GW}^U(M, U) = \inf R_{GW}^U(M, U),\]
where the infimum is over all choices of $U$ with $|U| \leq |X_1|+|X_2|+2$.

A. Optimality of GW-LFU-MN

In order to prove the optimality of the GW-LFU-MN scheme, we first state the following theorem:

Theorem 6: For any $U$, if
\[M \in \left[ \min \{H(X_1|U), H(X_2|U)\}, H(X_1, X_2) \right],\]
then $R_{GW}^U(M, U) = R_{GW}^U(M, U)$.

Proof 6: Similar to the proof of Theorem 3, $R_{GW}^U(M, U)$ is compared to the lower bound $R_{LB}^U(M, U)$ in each memory region.

Example 2: Assuming the same setting as in Example 1, since $R_{LB}^U(M) = R_{GW}^U(M)$ for any $M$, it follows from Theorem 6, that the GW-LFU-MN scheme is optimal for all values of memory in
\[M \in \left[ \min \{H(X_1|X_2), H(X_2|X_1)\}, H(X_1, X_2) \right].\]

From Theorems 1, 3, 5, and 6, after some algebra, we have:

Theorem 7: Let
\[\tilde{M}_1 = \max_{X_1 \sim U \sim X_2} \frac{1}{2} \min \left\{ H(X_1|U), H(X_2|U) \right\},\]
where the max is over all choices of $U$ such that $p(x_1|u) = p(x_2|u)$. Then, for $M \in [H(X_1, X_2) - 2\tilde{M}_1, H(X_1, X_2)]$, the GW-LFU-MN scheme is optimal i.e., $R_{GW}^U(M) = R_*(M)$.

In addition, for $M \in \left[0, \min \left\{ 2\tilde{M}_1, \frac{1}{2} \min \{H(X_1|X_2), H(X_2|X_1)\} \right\} \right]$, $R_{GW}^U(M) - R_*(M) \leq \frac{M}{2} \leq \frac{1}{2} \min \left\{ H(X_1|X_2), H(X_2|X_1) \right\}$. 

\(2\)LFU is a local caching policy that, in the setting of this paper, leads to all receivers caching the same part of the file.
Finally, for \( M \in [0, H(X_1, X_2)] \),
\[
R_{GW}^{UB}(M) - R^*(M) \leq \frac{1}{2} \min \{H(X_1|X_2), H(X_2|X_1)\}.
\]

Proof 7: See [14]

B. Illustration of Results: Doubly Symmetric Binary Source

Consider, as a 2-DMS, a doubly symmetric binary source (DSBS) with joint pmf \( p(x_1, x_2) = \frac{1}{2} (1-p_0) \delta_{x_1,x_2} + \frac{1}{2} p_0 (1-\delta_{x_1,x_2}) \), \( x_1, x_2 \in \{0,1\} \), and parameter \( p_0 \in [0, \frac{1}{2}] \). Then,
\[
\begin{align*}
H(X_1) &= H(X_2) = 1, \\
H(X_1|X_2) &= H(X_2|X_1) = h(p_0), \\
H(X_1, X_2) &= 1 + h(p_0),
\end{align*}
\]
where \( h(p) = -p \log(p) - (1-p) \log(1-p) \) is the binary entropy function. As derived in [12], the achievable Gray-Wyner rate region of a DSBS restricted to the plane \((R_0, R_1, R_2) : R_1 = R_2 = R \) is described by the set of rate triplets \((R_0, \rho, \rho)\) with \( R_0 \) given by
\[
R_0 \geq \begin{cases} 
1 + h(p_0) - 2\rho, & 0 \leq \rho < h(p_1) \\
f(\rho), & h(p_1) \leq \rho \leq 1,
\end{cases}
\]
where \( p_1 = \frac{1}{2} (1 - \sqrt{(1-2p_0)}) \),
\[
f(\rho) \triangleq 1 + h(p_0) + \left( h^{-1}(\rho) - \frac{p_0}{2} \right) \log \left( h^{-1}(\rho) - \frac{p_0}{2} \right) + p_0 \log \left( \frac{p_0}{2} \right) + \left( h^{-1}(\rho) - \frac{p_0}{2} \right) \log \left( h^{-1}(\rho) - \frac{p_0}{2} \right),
\]
and \( h^{-1}(\rho) \) is the inverse of the binary entropy function.

We compare the performance of the proposed GW-LFU-MN scheme with respect to: 1) LFU caching with uncoded multicasting (LFU-UM), 2) the deterministic correlation-unaware CACM in [1], referred to as MN, 3) the lower bound on the GW-CACM worst-case rate-memory function \( R_{GW}^{LB} \), and 4) the lower bound on the optimal worst-case rate-memory function \( R^{LB} \). Fig. 2 displays the rate-memory trade-offs for \( p_0 = 0.1 \). In line with Theorem 7, Fig. 2 shows that GW-LFU-MN meets the optimal worst-case rate-memory function \( R^*(M) \), for \( M \geq (H(X_1, X_2) - 2M_1) = 1.17 \), while correlation-unaware schemes, LFU-UM and MN, fall short. As stated in Theorem 3, the lower bound on the Gray-Wyner worst-case rate-memory function \( R_{GW}^{LB} \) coincides with the lower bound on the optimal worst-case rate-memory function \( R^{LB} \) for \( M \leq M_1 = 0.15 \) and \( M \geq (H(X_1, X_2) - 2M_1) = 1.17 \). Finally, in line with Theorem 6, GW-LFU-MN achieves \( R_{GW}^{UB} \) for \( M \geq 0.35 \).

VI. CONCLUDING REMARKS

In this paper, we have studied the fundamental limits of cache-aided communication systems under the assumption of correlated content for a two-user two-file network. We have derived a lower bound on the rate-memory function for such systems and proposed a class of schemes based on a two-step source coding approach. Files are first compressed using Gray-Wyner source coding, and then cached and delivered using a combination of existing correlation-unaware cached aided coded multicast schemes. We have fully characterized the rate-memory trade-off of such class of schemes, proposed an achievable two-step scheme, and proved its optimality for different memory regimes.

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