Reconstruction/Triangulation
Old book Ch11.1 F&P
New book Ch7.2 F&P

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CS-GY 6643, Spring 2016

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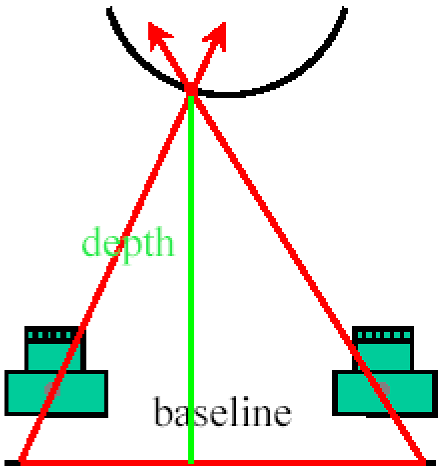
(modified from original slides by J. Ponce and by Marc Pollefeys)

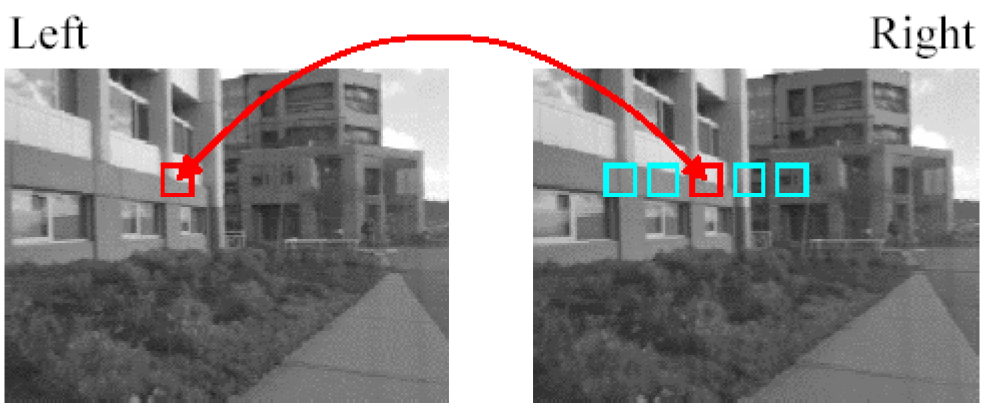
Credits: J. Ponce, M. Pollefeys, A. Zisserman & S. Lazebnik

Reconstruction



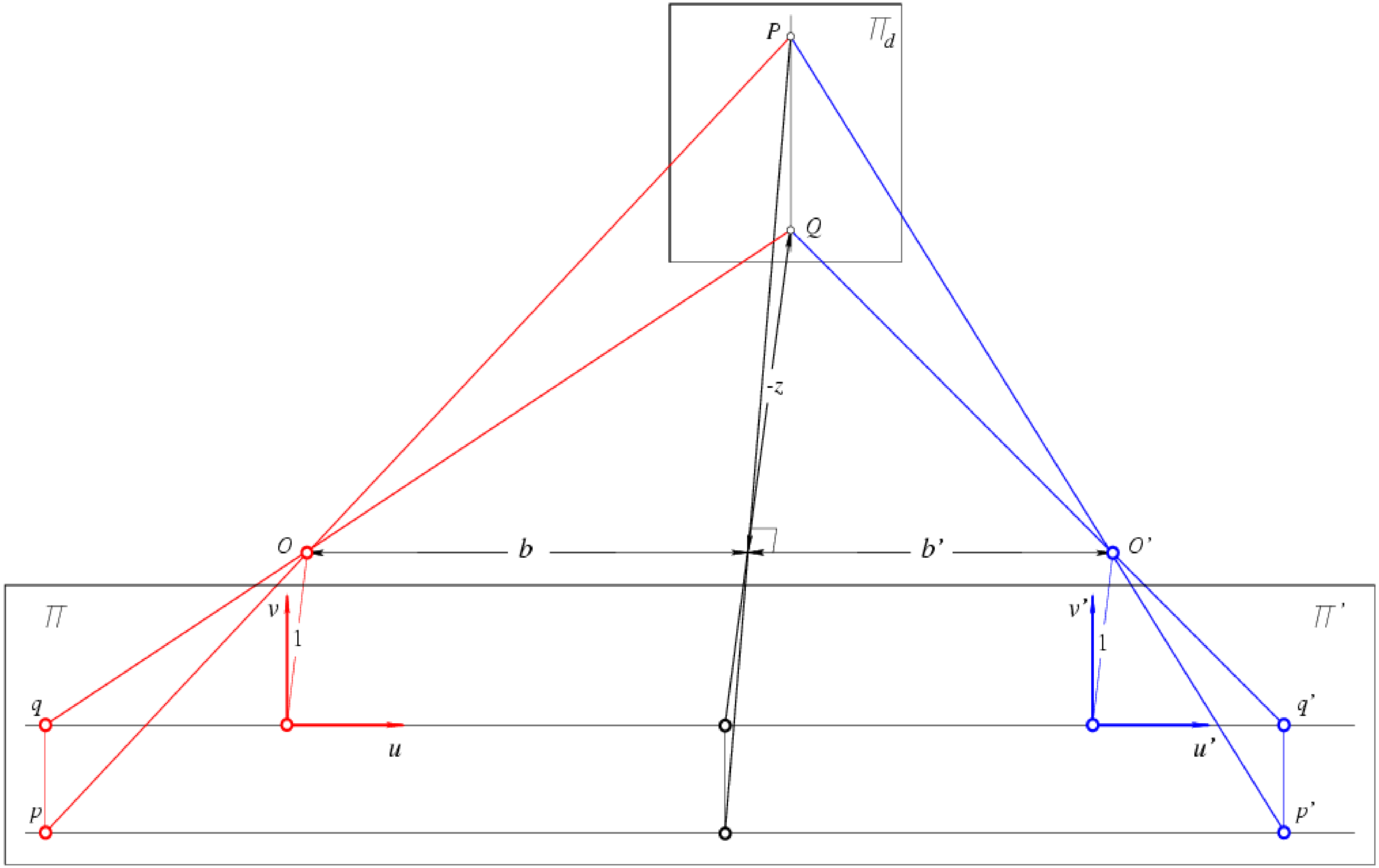
Triangulate on two images of the same point to recover depth.

- Feature matching across views
- Calibrated cameras



Only need to match features across epipolar lines

Reconstruction from Rectified Images

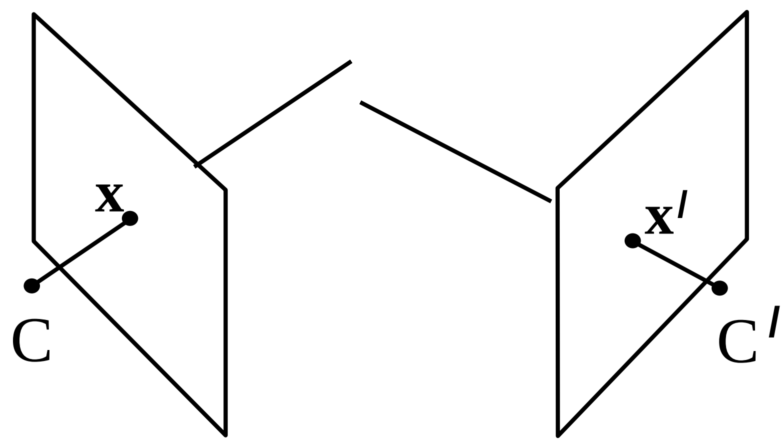


Disparity: $d=u'-u$ → Depth: $z = -Bf/d$

Problem statement

Given: corresponding measured (i.e. noisy) points $\mathbf{x}$ and $\mathbf{x}'$, and cameras (exact) P and P$'$, compute the 3D point $\mathbf{X}$

Problem: in the presence of noise, back projected rays do not intersect



rays are skew in space

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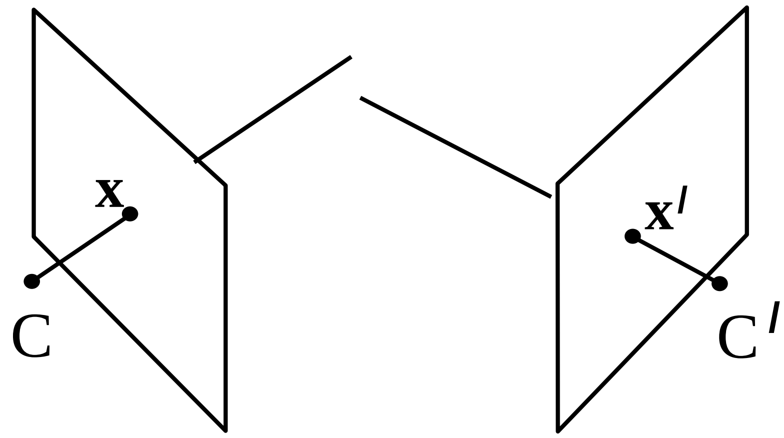
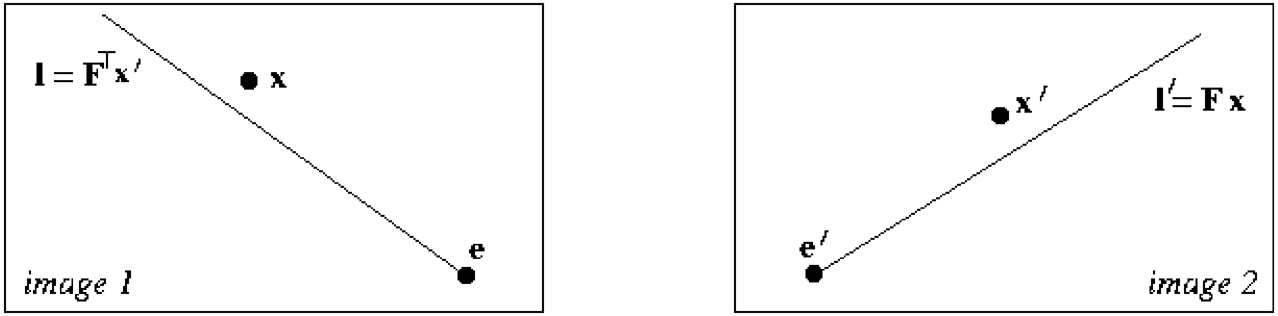
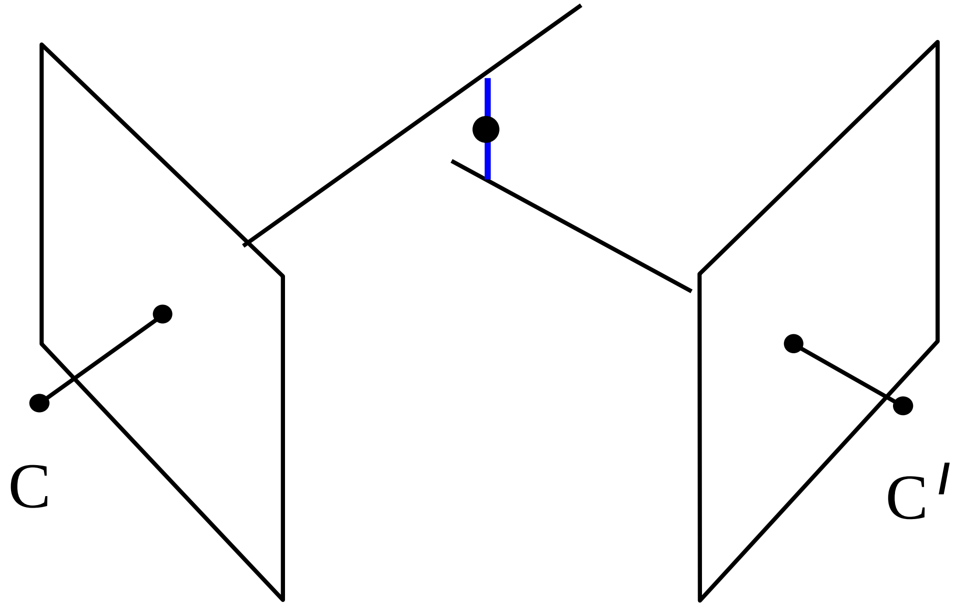


rays are skew in space



Measured points do not lie on corresponding epipolar lines

1. Vector solution

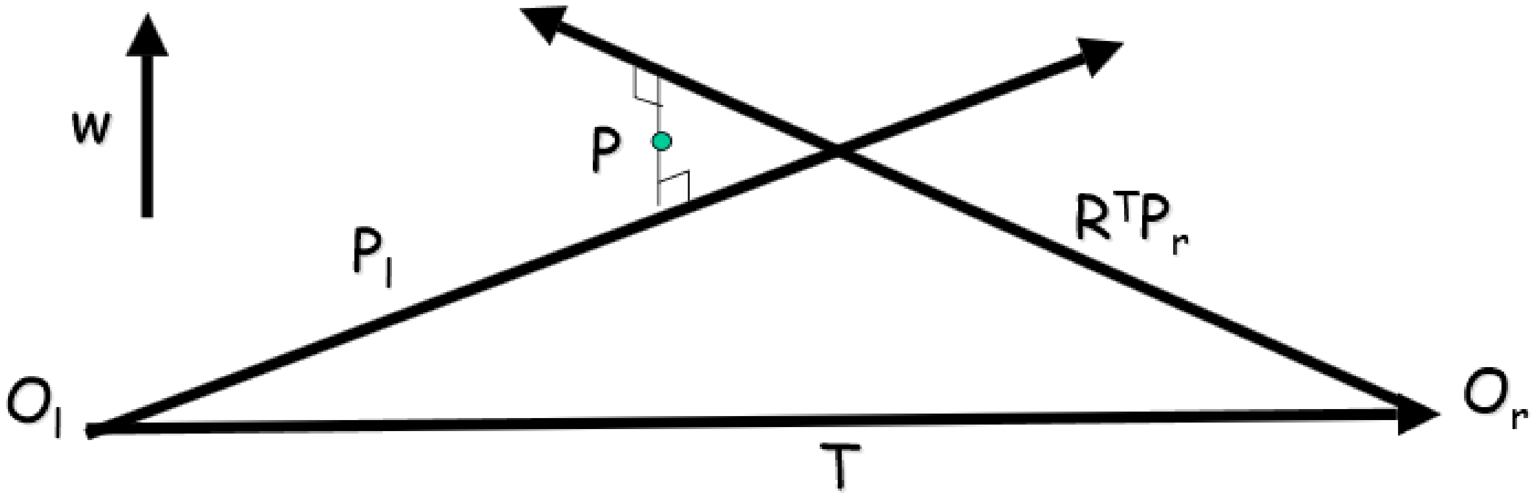


Compute the mid-point of the shortest line between the two rays

Solution from Trucco & Verri Book

P is midpoint of the segment perpendicular to P_l and $R^T P_r$

Let $w = P_l \times R^T P_r$ (this is perpendicular to both)



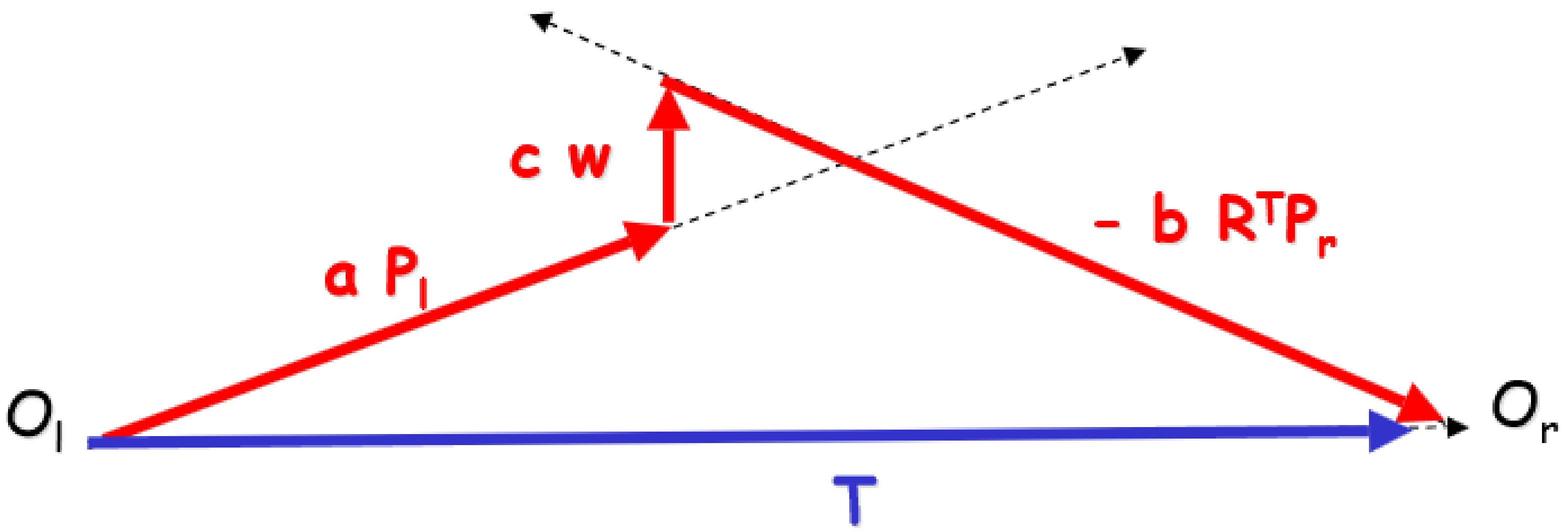
Introducing three unknown scale factors a,b,c we note we can write down the equation of a "circuit"

Source: Collins, CSE486 Penn State

Solution from Trucco & Verri Book

Writing vector "circuit diagram" with unknowns a,b,c

$$a\,P_l + c\,(P_l \times R^T P_r) - b\,R^T P_r = T$$

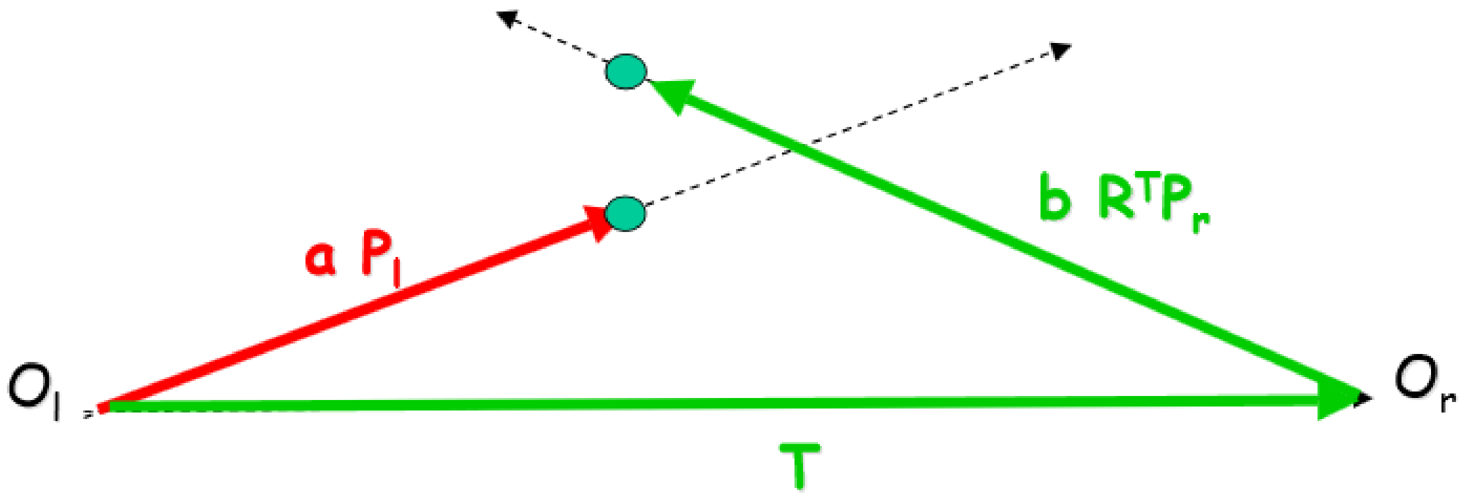


note: this is three linear equations in three unknowns a,b,c
=> can solve for a,b,c

Source: Collins, CSE486 Penn State

Solution from Trucco & Verri Book

After finding a,b,c , solve for midpoint of line segment between points $O_l + a\ P_l$ and $O_l + T + b\ R^T P_r$



Source: Collins, CSE486 Penn State

2. Linear triangulation (algebraic solution)

Use the equations $\mathbf{x} = \mathtt{P}\mathbf{X}$ and $\mathbf{x}' = \mathtt{P}'\mathbf{X}$ to solve for $\mathbf{X}$

For the first camera:

$$\mathtt{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} = \begin{bmatrix} \mathbf{p}^{1\top} \\ \mathbf{p}^{2\top} \\ \mathbf{p}^{3\top} \end{bmatrix}$$

where $\mathbf{p}^{i\top}$ are the rows of $\mathtt{P}$

- eliminate unknown scale in $\lambda\mathbf{x} = \mathtt{P}\mathbf{X}$ by forming a cross product $\mathbf{x} \times (\mathtt{P}\mathbf{X}) = \mathbf{0}$

$$\begin{aligned} x(\mathbf{p}^{3\top}\mathbf{X}) - (\mathbf{p}^{1\top}\mathbf{X}) &= 0 \\ y(\mathbf{p}^{3\top}\mathbf{X}) - (\mathbf{p}^{2\top}\mathbf{X}) &= 0 \\ x(\mathbf{p}^{2\top}\mathbf{X}) - y(\mathbf{p}^{1\top}\mathbf{X}) &= 0 \end{aligned}$$

- rearrange as (first two equations only)

$$\begin{bmatrix} x\mathbf{p}^{3\top} - \mathbf{p}^{1\top} \\ y\mathbf{p}^{3\top} - \mathbf{p}^{2\top} \end{bmatrix} \mathbf{X} = \mathbf{0}$$

Similarly for the second camera:

$$\begin{bmatrix} x'\mathbf{p}'^{3\top} - \mathbf{p}'^{1\top} \\ y'\mathbf{p}'^{3\top} - \mathbf{p}'^{2\top} \end{bmatrix} \mathbf{X} = \mathbf{0}$$

Collecting together gives

$$\mathtt{A}\mathbf{X} = \mathbf{0}$$

where $\mathtt{A}$ is the 4×4 matrix

$$\mathtt{A} = \begin{bmatrix} x\mathbf{p}^{3\top} - \mathbf{p}^{1\top} \\ y\mathbf{p}^{3\top} - \mathbf{p}^{2\top} \\ x'\mathbf{p}'^{3\top} - \mathbf{p}'^{1\top} \\ y'\mathbf{p}'^{3\top} - \mathbf{p}'^{2\top} \end{bmatrix}$$

from which $\mathbf{X}$ can be solved up to scale.

Problem: does not minimize anything meaningful

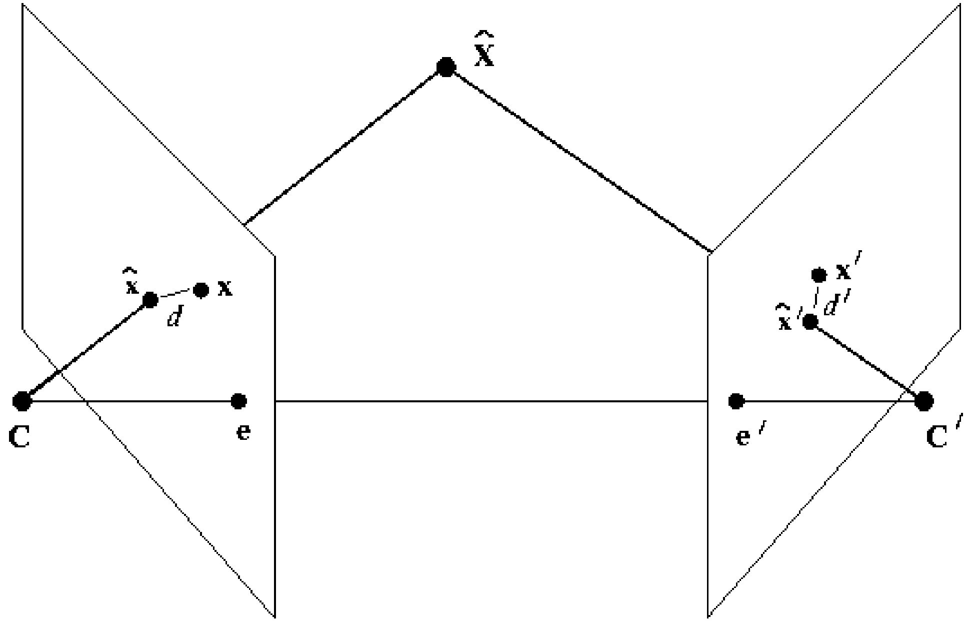
Advantage: extends to more than two views

3. Minimizing a geometric/statistical error

The idea is to estimate a 3D point $\hat{\mathrm{X}}$ which exactly satisfies the supplied camera geometry, so it projects as

$$\hat{\mathrm{x}} = \mathrm{P}\hat{\mathrm{X}} \qquad \hat{\mathrm{x}}' = \mathrm{P}'\hat{\mathrm{X}}$$

and the aim is to estimate $\hat{\mathrm{X}}$ from the image measurements $\mathbf{x}$ and $\mathbf{x}'$.



$$\min_{\hat{\mathrm{X}}} \quad \mathcal{C}(\mathbf{x}, \mathbf{x}') = d(\mathbf{x}, \hat{\mathbf{x}})^2 + d(\mathbf{x}', \hat{\mathbf{x}}')^2$$

where $d(*, *)$ is the Euclidean distance between the points.

- It can be shown that if the measurement noise is Gaussian mean zero, $\sim N(0, \sigma^2)$, then minimizing geometric error is the Maximum Likelihood Estimate of X

- The minimization appears to be over three parameters (the position X), but the problem can be reduced to a minimization over one parameter

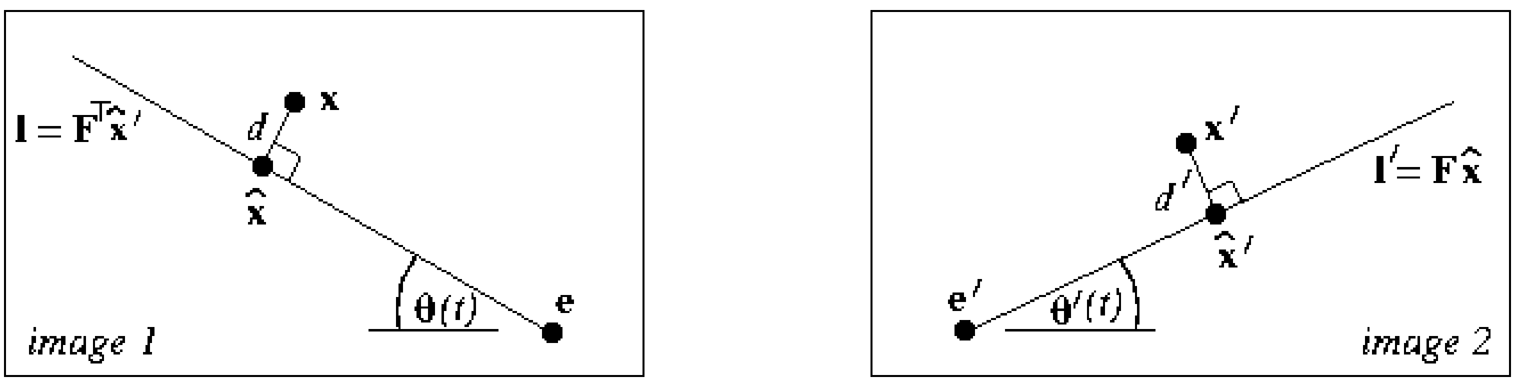
Different formulation of the problem

The minimization problem may be formulated differently:

- Minimize

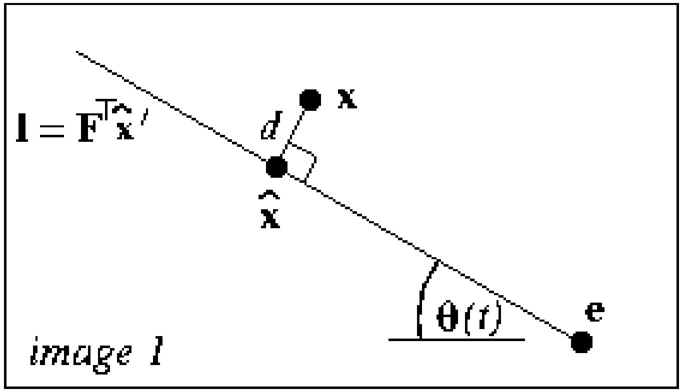
$$d(\mathbf{x}, \mathbf{l})^2 + d(\mathbf{x}', \mathbf{l}')^2$$

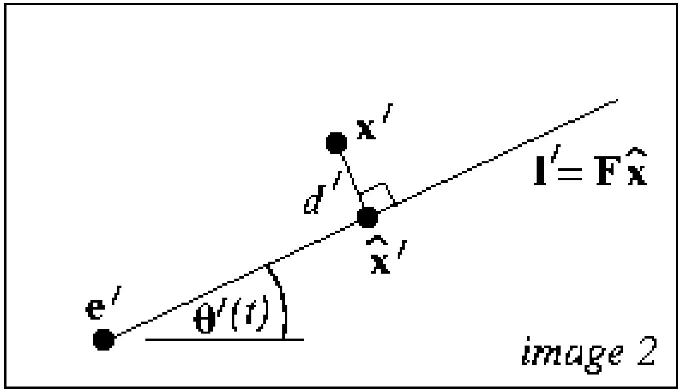
- $\mathbf{l}$ and $\mathbf{l}'$ range over all choices of corresponding epipolar lines.
- $\hat{\mathbf{x}}$ is the closest point on the line $\mathbf{l}$ to $\mathbf{x}$.
- Same for $\hat{\mathbf{x}}'$.



Minimization method

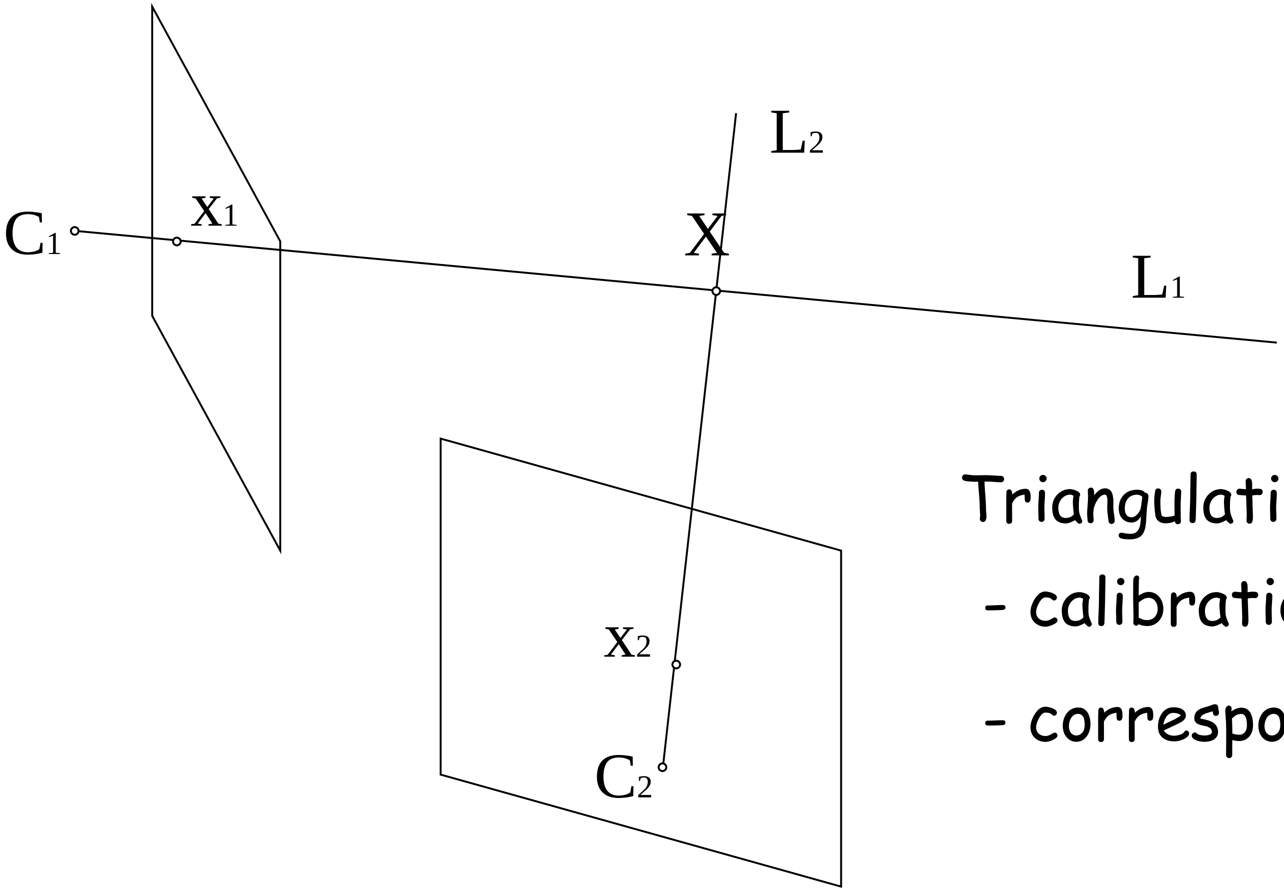
- Parametrize the pencil of epipolar lines in the first image by t, such that the epipolar line is $\mathbf{l}(t)$
- Using F compute the corresponding epipolar line in the second image $\mathbf{l}'(t)$
- Express the distance function $d(\mathbf{x}, \mathbf{l})^2 + d(\mathbf{x}', \mathbf{l}')^2$ explicitly as a function of t
- Find the value of t that minimizes the distance function
- Solution is a 6th degree polynomial in t





More slides for self-study.

Triangulation (finally!)



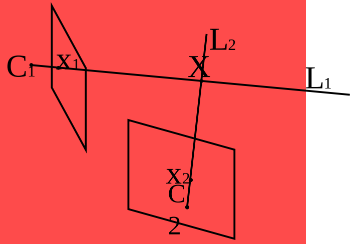
Triangulation

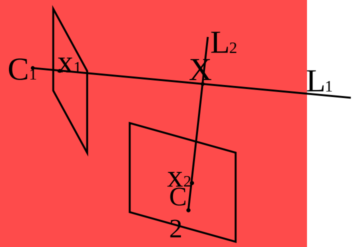
- calibration
- correspondences

Triangulation

- Backprojection

$$\lambda \mathrm{x} = \mathbf{P}\mathrm{X}$$





Triangulation

- Backprojection

$$\lambda \mathrm{x} = \mathbf{P}\mathrm{X}$$

$$\begin{bmatrix} \lambda x \\ \lambda y \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathrm{P}_1 \\ \mathrm{P}_2 \\ \mathrm{P}_3 \end{bmatrix} \mathrm{X}$$

Triangulation

- Backprojection

$$\lambda \mathbf{x} = \mathbf{P}\mathtt{X}$$

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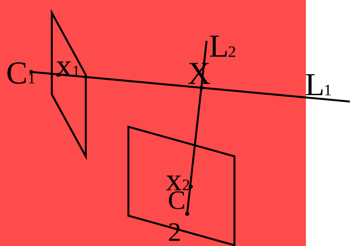
$$\begin{aligned} \mathtt{P}_3 \mathtt{X} x &= \mathtt{P}_1 \mathtt{X} \\ \mathtt{P}_3 \mathtt{X} y &= \mathtt{P}_2 \mathtt{X} \end{aligned}$$

$$\begin{bmatrix} \mathtt{P}_3 x - \mathtt{P}_1 \\ \mathtt{P}_3 y - \mathtt{P}_2 \end{bmatrix} \mathtt{X} = 0$$



Triangulation

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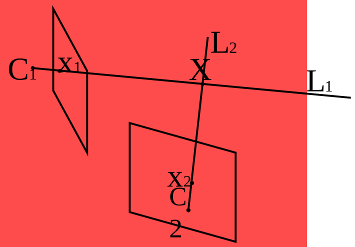
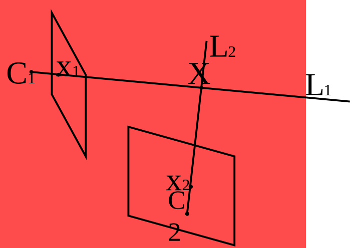


- Triangulation

$$\begin{bmatrix} \mathtt{P}_3 x - \mathtt{P}_1 \\ \mathtt{P}_3 y - \mathtt{P}_2 \\ \mathtt{P}'_3 x' - \mathtt{P}'_1 \\ \mathtt{P}'_3 y' - \mathtt{P}'_2 \end{bmatrix} \mathtt{X} = 0$$

Triangulation



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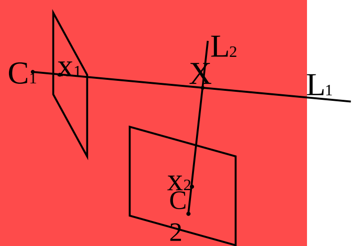
$$\begin{bmatrix} \frac{1}{\mathbf{P}_3\tilde{\mathbf{X}}} \begin{pmatrix} \mathbf{P}_3 x - \mathbf{P}_1 \\ \mathbf{P}_3 y - \mathbf{P}_2 \end{pmatrix} \\ \frac{1}{\mathbf{P}'_3\tilde{\mathbf{X}}} \begin{pmatrix} \mathbf{P}'_3 x - \mathbf{P}'_1 \\ \mathbf{P}'_3 y - \mathbf{P}'_2 \end{pmatrix} \end{bmatrix} \mathbf{X} = 0$$

Iterative least-squares



Triangulation

- Backprojection

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- Triangulation

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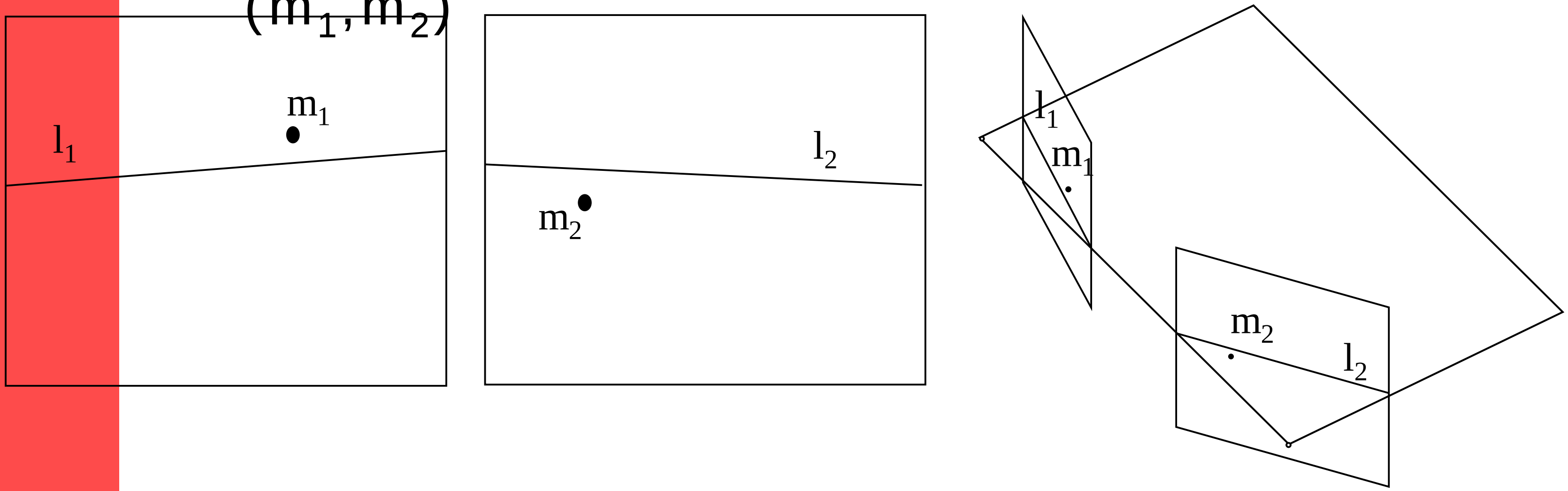
Iterative least-squares

- Maximum Likelihood Triangulation

$$\arg\min_{\mathrm{X}} \sum_i \left(\mathrm{x}_i - \lambda^{-1} \mathbf{P}_i \mathrm{X} \right)^2$$

Optimal 3D point in epipolar plane

- Given an epipolar plane, find best 3D point for (m_1,m_2)



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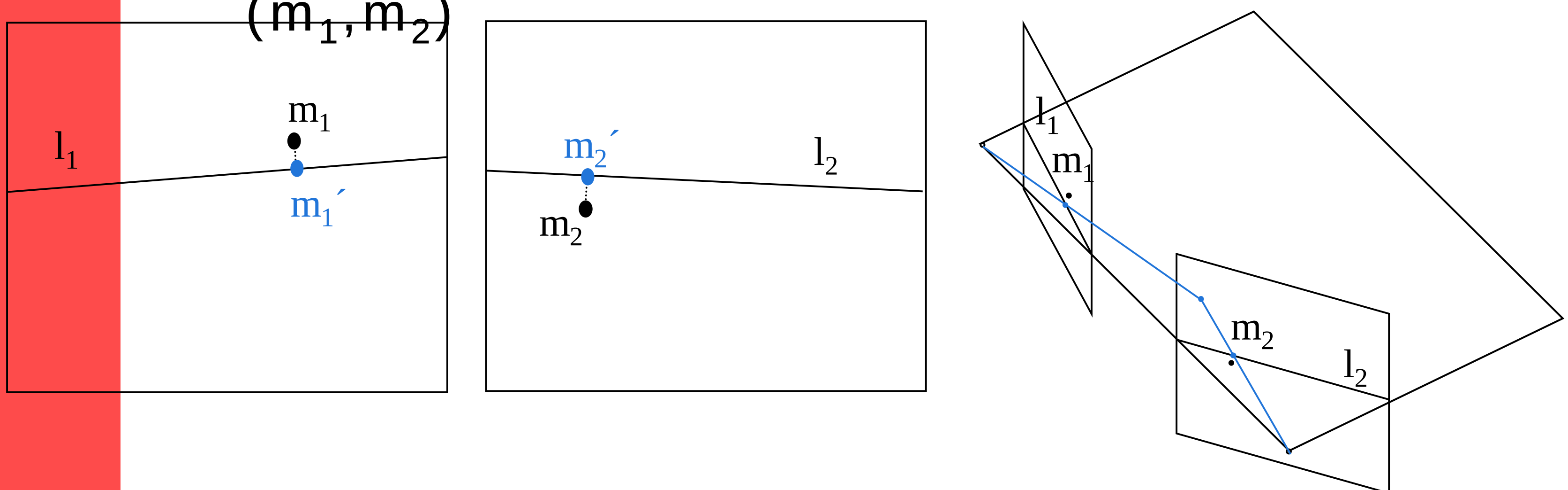


Select closest points $(m_1´, m_2´)$ on epipolar lines

Obtain 3D point through exact triangulation

Guarantees minimal reprojection error (given this epipolar plane)

Non-iterative optimal solution

- Reconstruct matches in projective frame by minimizing the reprojection error

$$D(\mathbf{m}_1,\mathbf{P}_1\mathbf{M})^2 + D(\mathbf{m}_2,\mathbf{P}_2\mathbf{M})^2 \quad \textbf{3DOF}$$

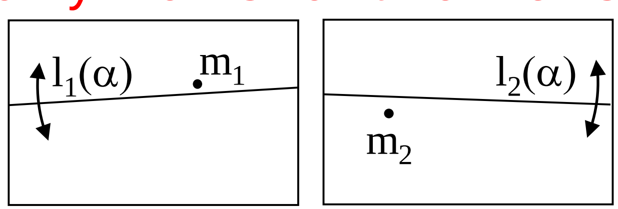
- Non-iterative method

  Determine the epipolar plane for reconstruction (Hartley and Sturm, CVIU´97)

$$D(\mathbf{m}_1,\mathbf{l}_1(\alpha))^2 + D(\mathbf{m}_2,\mathbf{l}_2(\alpha))^2 \text{ (polynomial of degree 6)}$$

  Reconstruct optimal point from selected epipolar plane

  Note: only works for two views



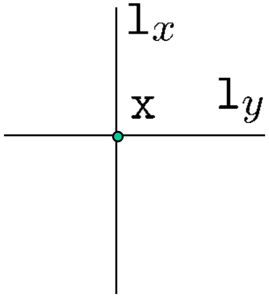
1DOF

Backprojection

- Represent point as intersection of row and column

$$\mathtt{x} = \mathtt{l}_x \times \mathtt{l}_y \text{ with } \mathtt{l}_x = \begin{bmatrix} -1 \\ 0 \\ x \end{bmatrix}, \mathtt{l}_y = \begin{bmatrix} 0 \\ -1 \\ y \end{bmatrix}$$

$$\Pi = \mathtt{P}^\top \mathtt{l}$$



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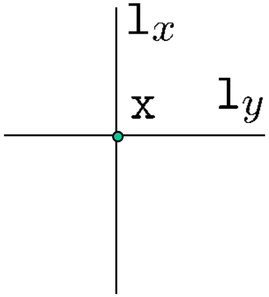

$$\Pi = \mathtt{P}^\top \mathtt{l}$$

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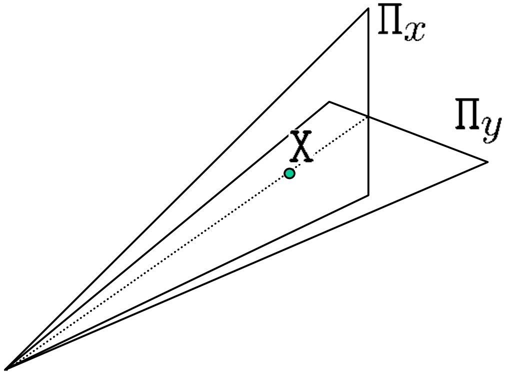

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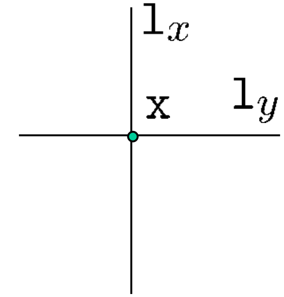

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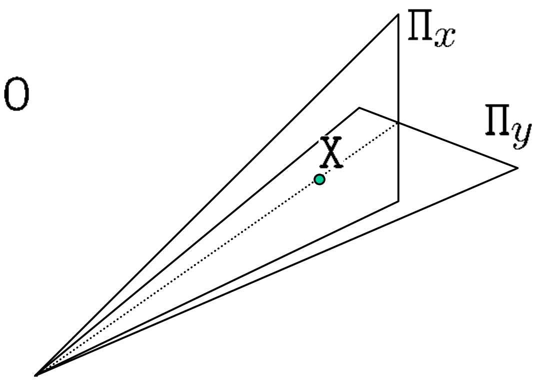
- Condition for solution?



Backprojection

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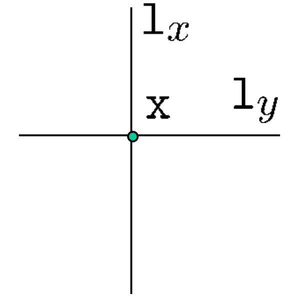

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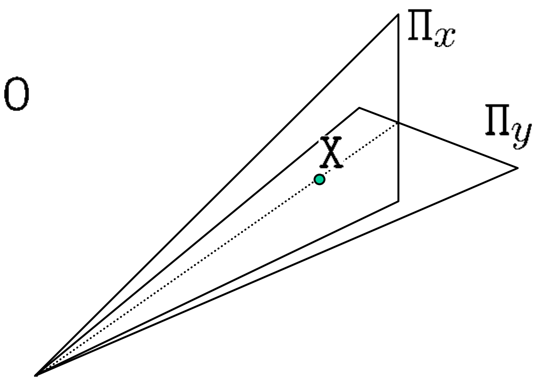
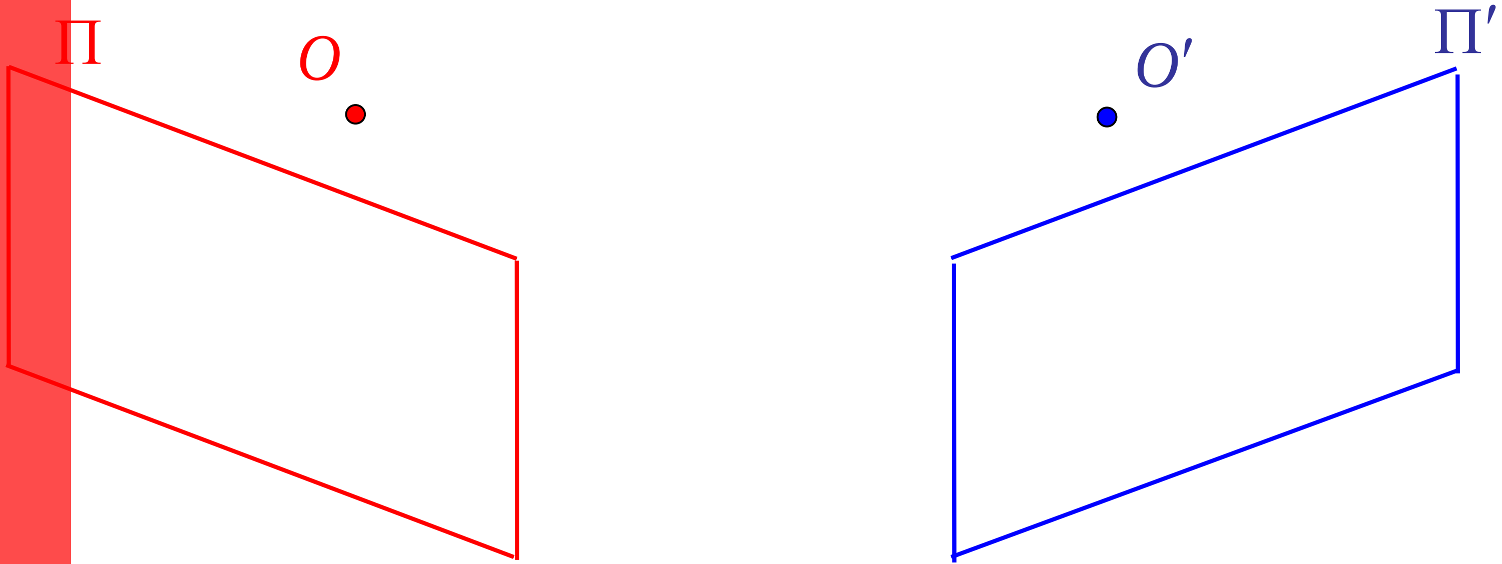


- Condition for solution?

$$\det \begin{bmatrix} \mathtt{l}_x^\top \mathtt{P} \\ \mathtt{l}_y^\top \mathtt{P} \\ \mathtt{l}_{x'}^\top \mathtt{P}' \\ \mathtt{l}_{y'}^\top \mathtt{P}' \end{bmatrix} = 0$$

Useful presentation for deriving and understanding multiple view geometry (notice 3D planes are linear in 2D point coordinates)

Geometric Reconstruction



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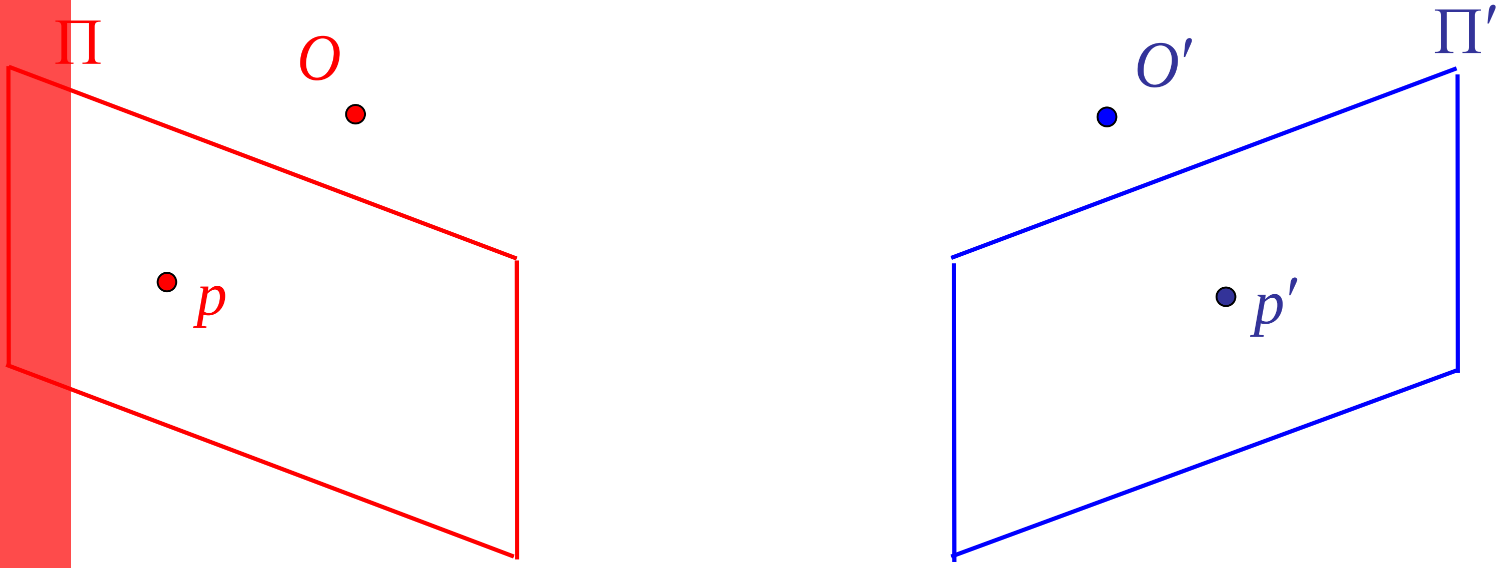

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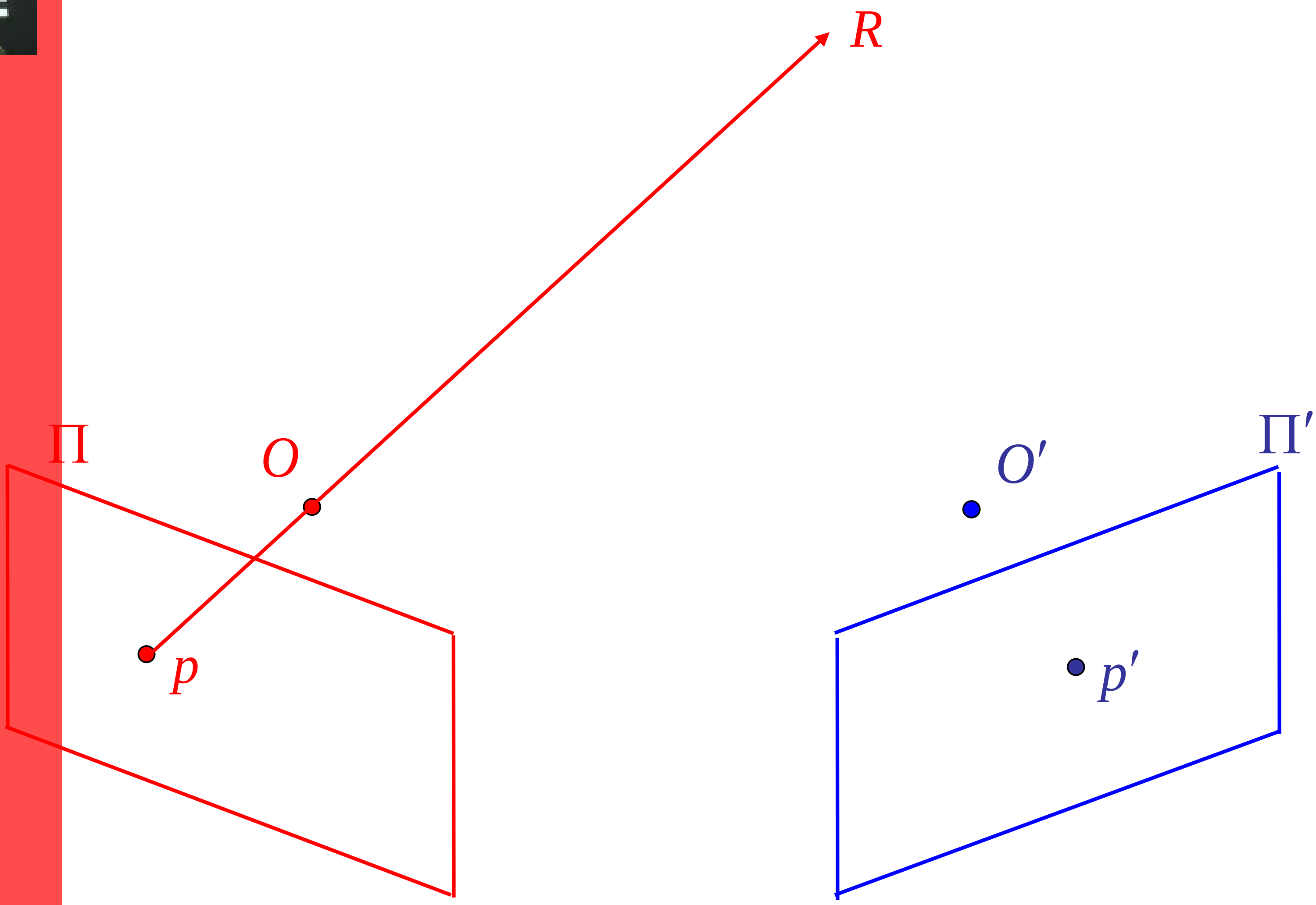

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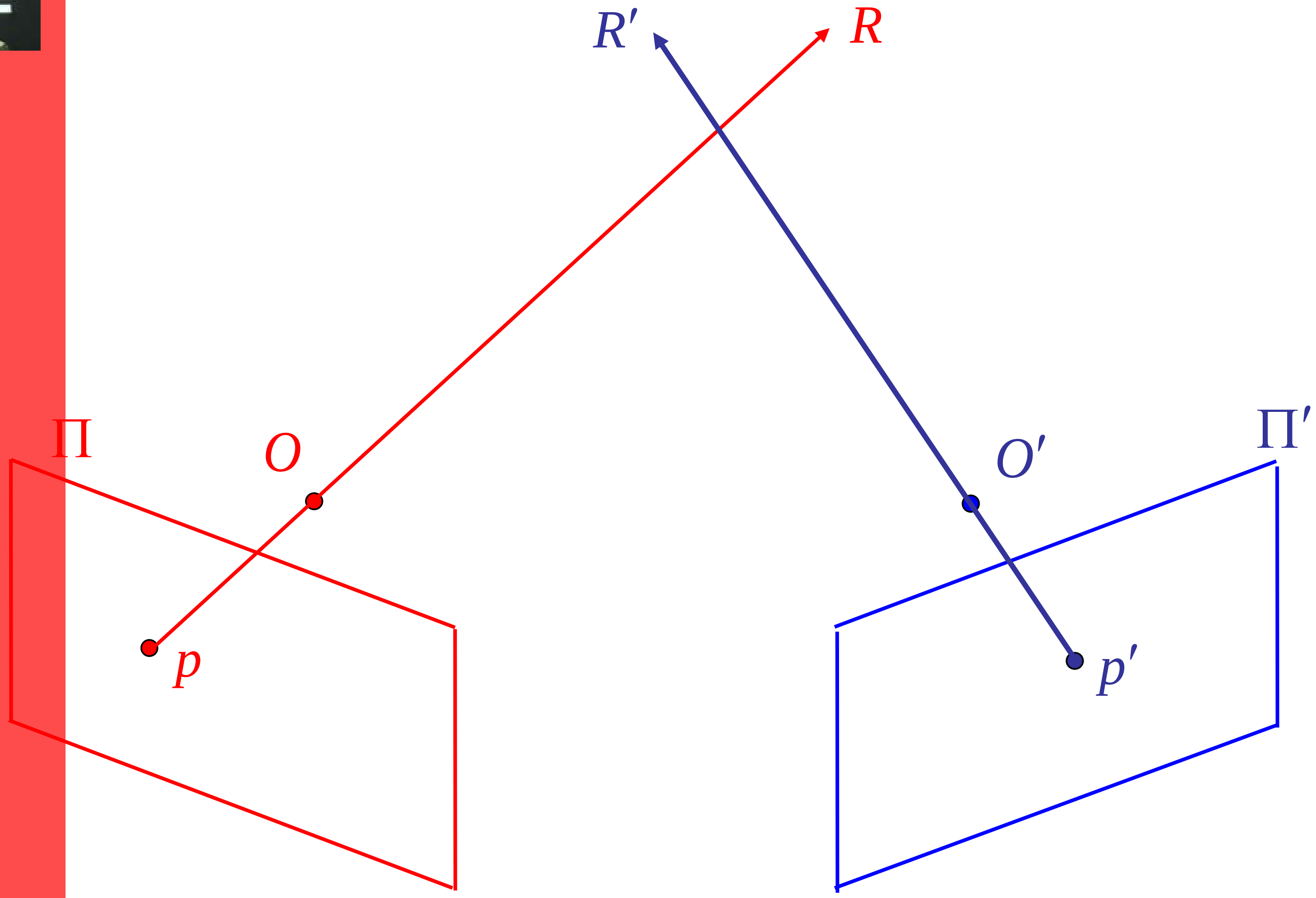
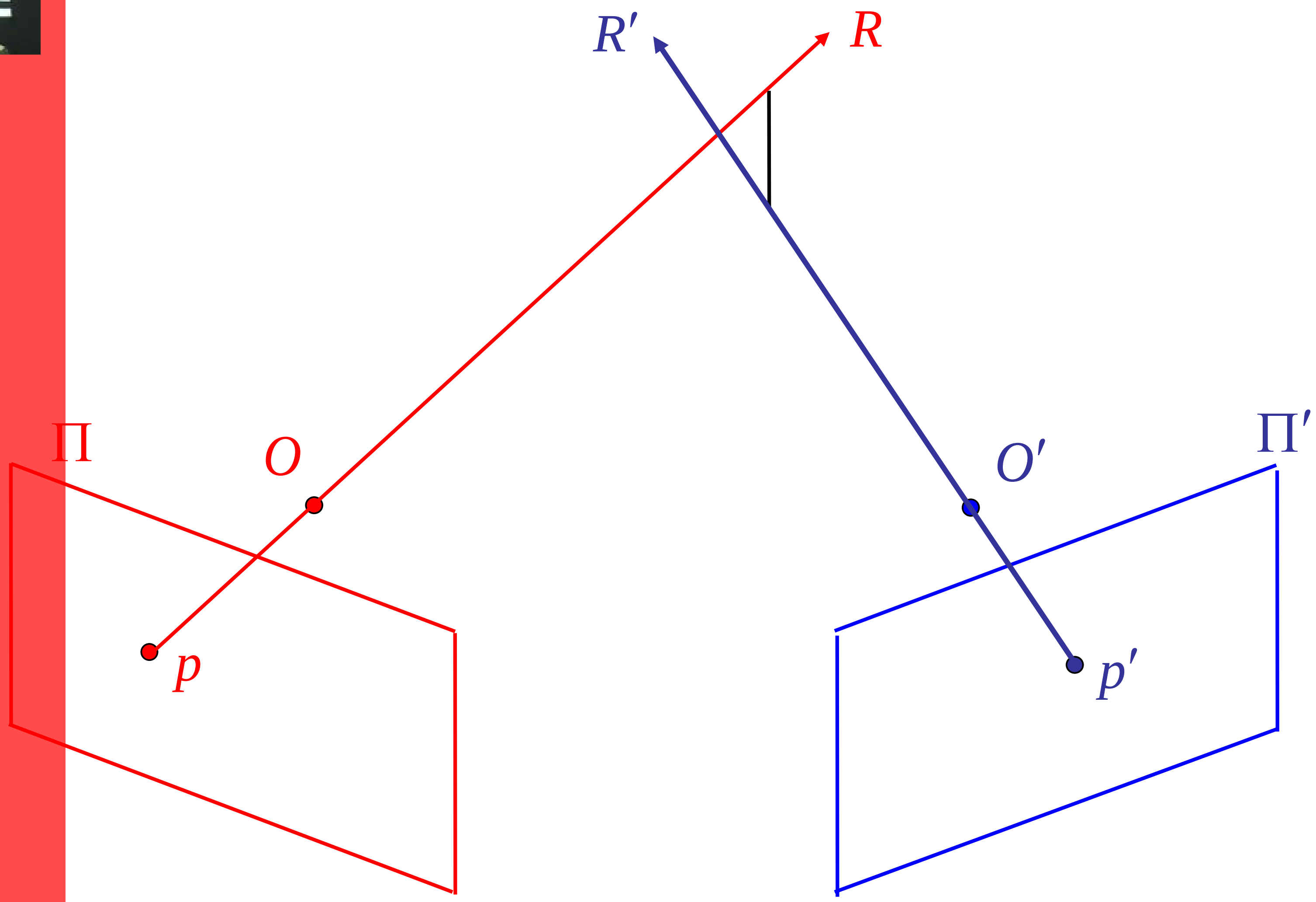
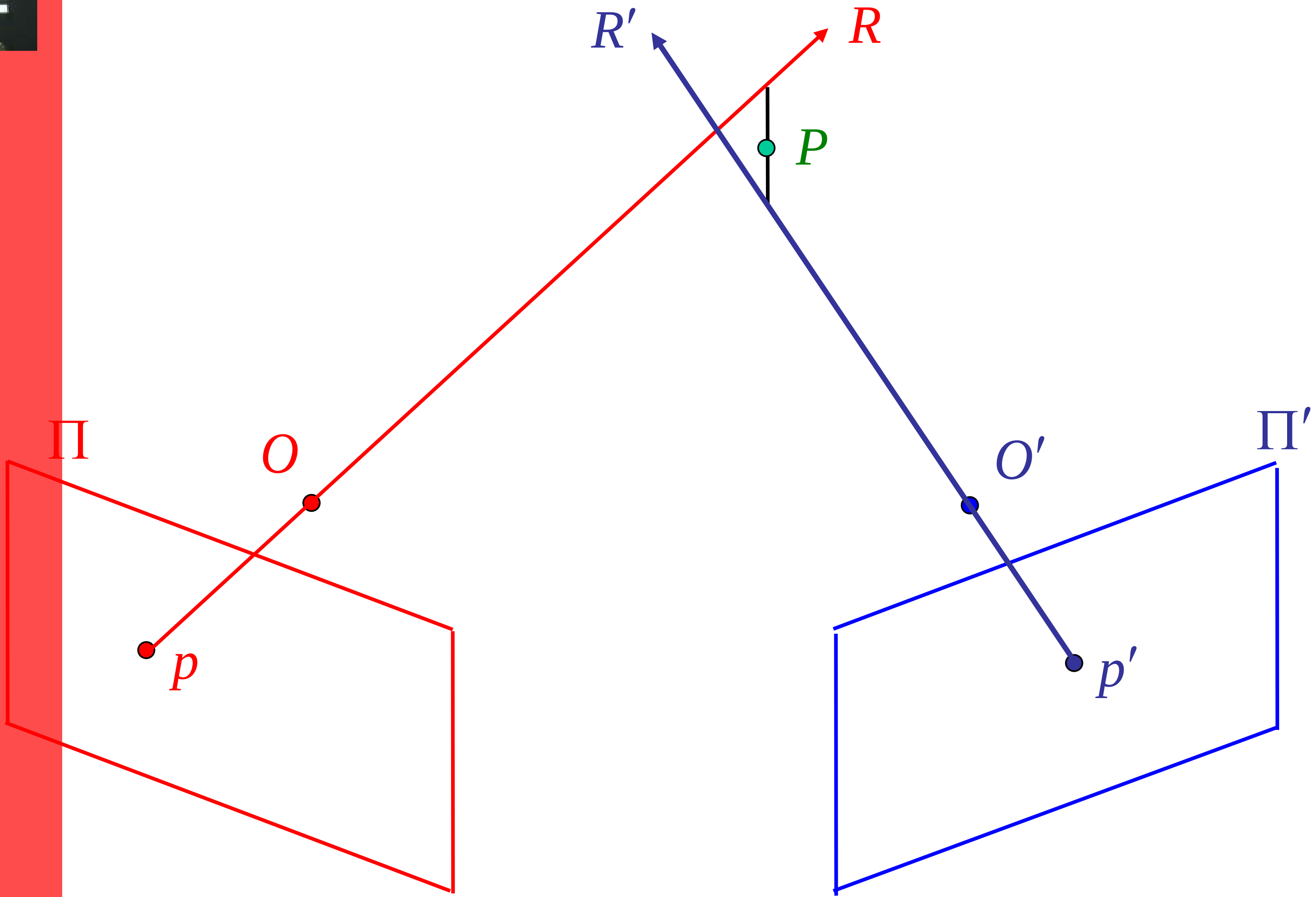


Geometric Reconstruction



Geometric Reconstruction



Reconstruction

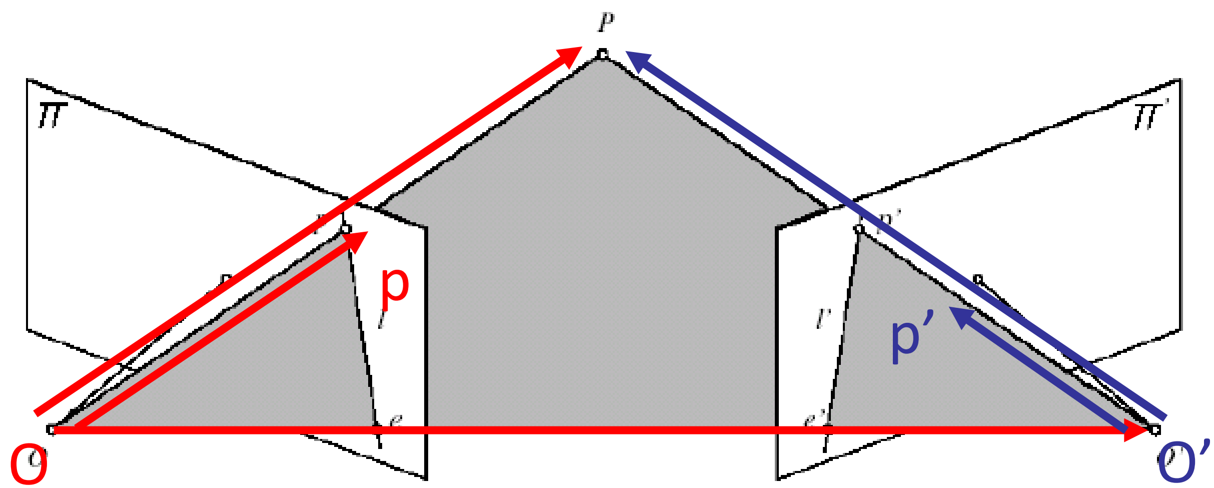


FIGURE 11.1: Epipolar geometry: the point P, the optical centers O and O' of the two cameras, and the two images p and p' of P all lie in the same plane.

Reconstruction

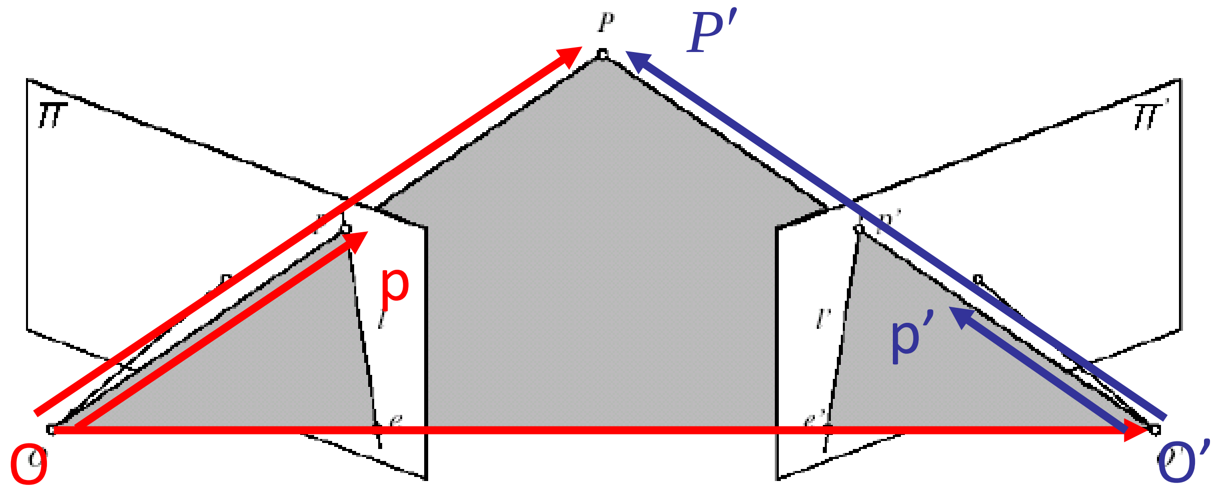


FIGURE 11.1: Epipolar geometry: the point P, the optical centers O and O' of the two cameras, and the two images p and p' of P all lie in the same plane.

Reconstruction

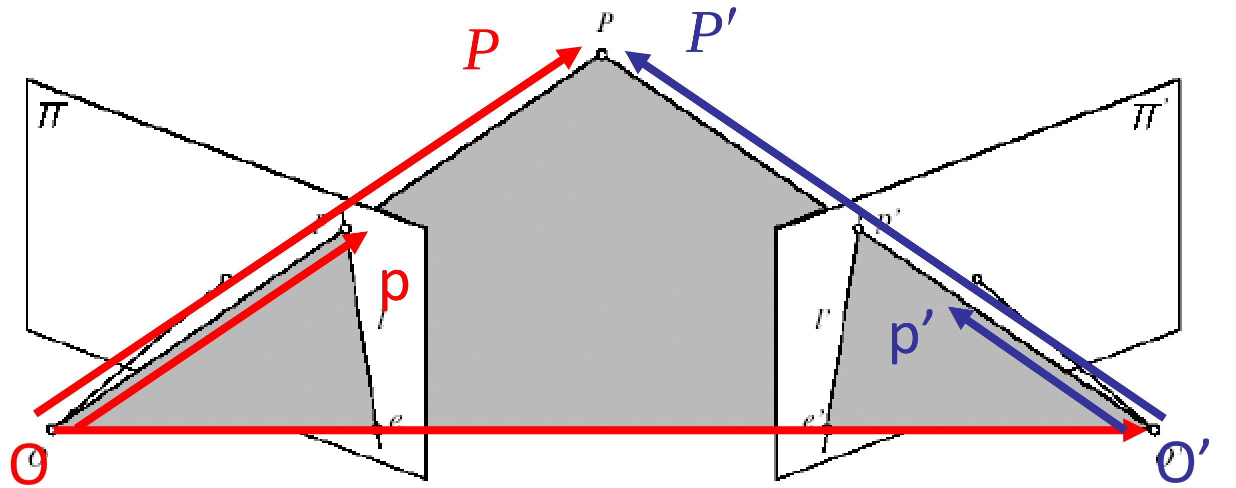
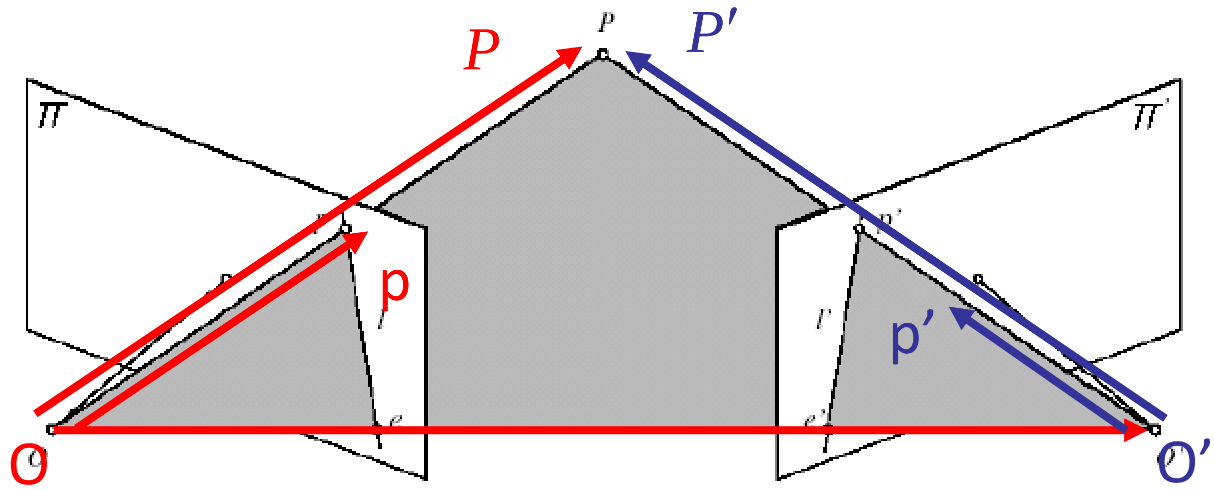


FIGURE 11.1: Epipolar geometry: the point P, the optical centers O and O' of the two cameras, and the two images p and p' of P all lie in the same plane.

Reconstruction

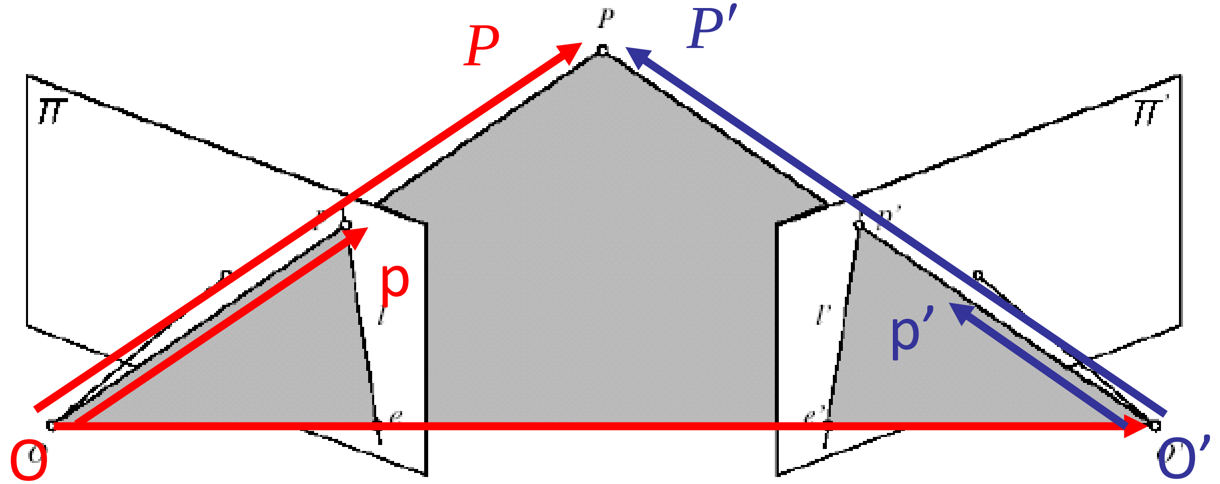


$$p' = f' \frac{P'}{Z'}$$

FIGURE 11.1: Epipolar geometry: the point P, the optical centers O and O' of the two cameras, and the two images p and p' of P all lie in the same plane.

Reconstruction

$$p = f\frac{P}{Z}$$

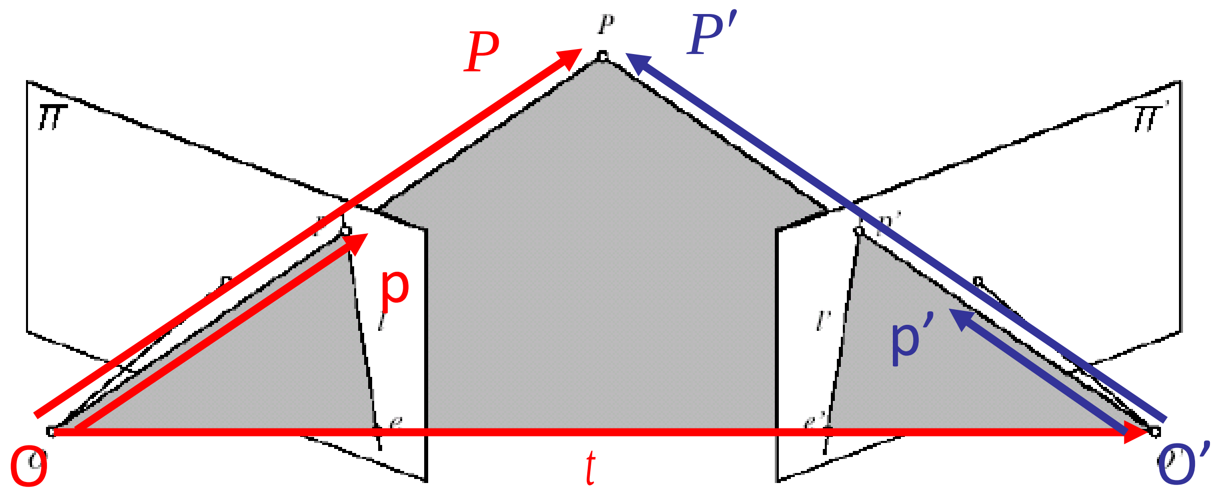


$$p' = f'\frac{P'}{Z'}$$

FIGURE 11.1: Epipolar geometry: the point P, the optical centers O and O' of the two cameras, and the two images p and p' of P all lie in the same plane.

Reconstruction

$$p = f\,\frac{P}{Z}$$

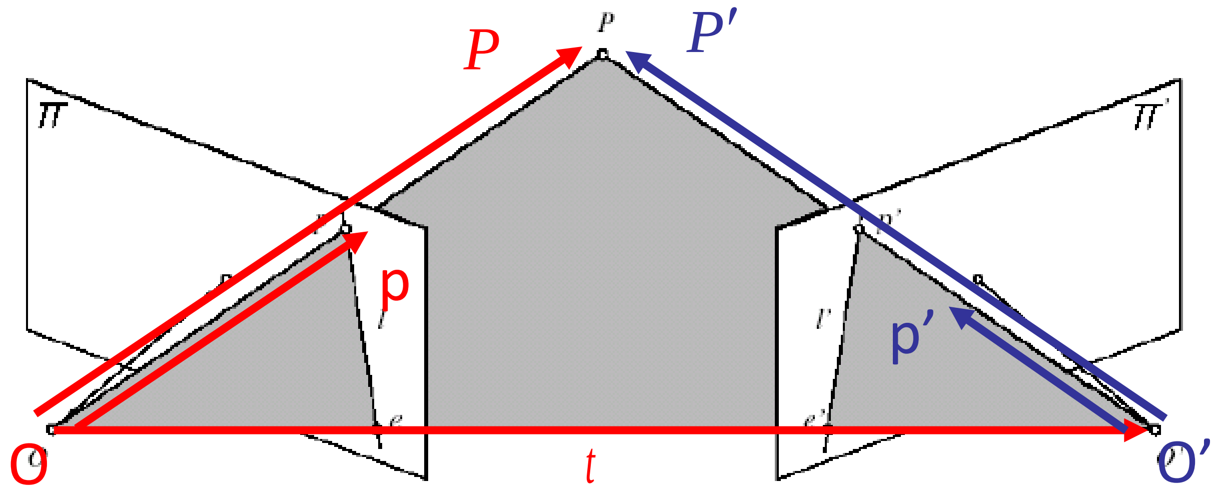


$$p' = f'\,\frac{P'}{Z'}$$

FIGURE 11.1: Epipolar geometry: the point P, the optical centers O and O' of the two cameras, and the two images p and p' of P all lie in the same plane.

Reconstruction

$$p = f\frac{P}{Z}$$



$$p' = f'\frac{P'}{Z'}$$

FIGURE 11.1: Epipolar geometry: the point P, the optical centers O and O' of the two cameras, and the two images p and p' of P all lie in the same plane.

$$P = RP' + t$$

$$P' = R^{-1}(P - t) = R^{T}(P - t)$$

Reconstruction

Reconstruction

$$p' = f' \frac{P'}{Z'}$$

Reconstruction

$$p' = f' \frac{P'}{Z'}$$

$$P' = R^T (P - t) = R'(P - t)$$

Reconstruction

$$p' = f' \frac{P'}{Z'}$$

$$P' = R^T (P - t) = R'(P - t)$$

$$R' = \begin{bmatrix} R_1'^T \\ R_2'^T \\ R_3'^T \end{bmatrix}$$

Reconstruction

$$p' = f' \frac{P'}{Z'}$$

$$P' = R^T(P - t) = R'(P - t)$$

$$p' = f' \frac{R'(P - t)}{R_3'^T(P - t)}$$

$$R' = \begin{bmatrix} R_1'^T \\ R_2'^T \\ R_3'^T \end{bmatrix}$$

Reconstruction

$$p' = f' \frac{P'}{Z'}$$

$$P' = R^T(P-t) = R'(P-t)$$

$$p' = f' \frac{R'(P-t)}{R_3'^T(P-t)}$$

$$x' = f' \frac{R_1'^T(P-t)}{R_3'^T(P-t)}$$

$$R' = \begin{bmatrix} R_1'^T \\ R_2'^T \\ R_3'^T \end{bmatrix}$$

Reconstruction

$$p' = f' \frac{P'}{Z'}$$

$$P' = R^T (P - t) = R'(P - t)$$

$$R' = \begin{bmatrix} R_1'^T \\ R_2'^T \\ R_3'^T \end{bmatrix}$$

$$p' = f' \frac{R'(P - t)}{R_3'^T (P - t)}$$

$$x' = f' \frac{R_1'^T (P - t)}{R_3'^T (P - t)}$$ —— Equation 1

Reconstruction

$$p' = f' \frac{P'}{Z'}$$

$$P' = R^T (P - t) = R'(P - t)$$

$$R' = \begin{bmatrix} R_1'^T \\ R_2'^T \\ R_3'^T \end{bmatrix}$$

$$p' = f' \frac{R'(P - t)}{R_3'^T (P - t)}$$

$$x' = f' \frac{R_1'^T (P - t)}{R_3'^T (P - t)}$$ Equation 1

$$p = f \frac{P}{Z}$$

Reconstruction

$$p' = f' \frac{P'}{Z'}$$

$$P' = R^T(P - t) = R'(P - t)$$

$$R' = \begin{bmatrix} R_1'^T \\ R_2'^T \\ R_3'^T \end{bmatrix}$$

$$p' = f' \frac{R'(P - t)}{R_3'^T(P - t)}$$

$$x' = f' \frac{R_1'^T(P - t)}{R_3'^T(P - t)}$$ —— Equation 1

$$p = f \frac{P}{Z} \Rightarrow P = \frac{pZ}{f}$$

Reconstruction

$$p' = f'\frac{P'}{Z'}$$

$$P' = R^T(P-t) = R'(P-t)$$

$$R' = \begin{bmatrix} R_1'^T \\ R_2'^T \\ R_3'^T \end{bmatrix}$$

$$p' = f'\frac{R'(P-t)}{R_3'^T(P-t)}$$

$$x' = f'\frac{R_1'^T(P-t)}{R_3'^T(P-t)}$$ Equation 1

$$p = f\frac{P}{Z} \Rightarrow P = \frac{pZ}{f}$$ Equation 2

Reconstruction

$$p' = f' \frac{P'}{Z'}$$

$$P' = R^T(P-t) = R'(P-t)$$

$$R' = \begin{bmatrix} R_1'^T \\ R_2'^T \\ R_3'^T \end{bmatrix}$$

$$p' = f' \frac{R'(P-t)}{R_3'^T(P-t)}$$

$$x' = f' \frac{R_1'^T(P-t)}{R_3'^T(P-t)}$$ ——— Equation 1

$$p = f\frac{P}{Z} \Rightarrow P = \frac{pZ}{f}$$ ——— Equation 2

$$Z = f\frac{(x'R_3' - f'R_1')^T t}{(x'R_3' - f'R_1')^T p}$$

(From equations 1 and 2)

Reconstruction up to a Scale Factor

- Assume that intrinsic parameters of both cameras are known
- Essential Matrix is known up to a scale factor (for example,
estimated from the 8 point algorithm).

Reconstruction up to a Scale Factor

$$\mathcal{E} = k[t_\times]R$$

Reconstruction up to a Scale Factor

$$\mathcal{E} = k[t_\times]R$$

$$\mathcal{E}\mathcal{E}^T$$

Reconstruction up to a Scale Factor

$$\mathcal{E} = k[t_\times]R$$

$$\mathcal{E}\mathcal{E}^T = k^2[t_\times]RR^T[t_\times]^T$$

Reconstruction up to a Scale Factor

$$\mathcal{E} = k[t_\times]R$$

$$\mathcal{E}\mathcal{E}^T = k^2[t_\times]RR^T[t_\times]^T = k^2[t_\times][t_\times]^T$$

Reconstruction up to a Scale Factor

$$\mathcal{E} = k[t_\times]R$$

$$\mathcal{E}\mathcal{E}^T = k^2[t_\times]RR^T[t_\times]^T = k^2[t_\times][t_\times]^T = \begin{bmatrix} k^2\left(T_Y^2 + T_Z^2\right) & -k^2T_XT_Y & -k^2T_XT_Z \\ -k^2T_XT_Y & k^2\left(T_X^2 + T_Z^2\right) & -k^2T_YT_Z \\ -k^2T_XT_Z & -k^2T_YT_Z & k^2\left(T_X^2 + T_Y^2\right) \end{bmatrix}$$

Reconstruction up to a Scale Factor

$$\mathcal{E} = k[t_\times]R$$

$$\mathcal{E}\mathcal{E}^T = k^2[t_\times]RR^T[t_\times]^T = k^2[t_\times][t_\times]^T = \begin{bmatrix} k^2(T_Y^2+T_Z^2) & -k^2T_XT_Y & -k^2T_XT_Z \\ -k^2T_XT_Y & k^2(T_X^2+T_Z^2) & -k^2T_YT_Z \\ -k^2T_XT_Z & -k^2T_YT_Z & k^2(T_X^2+T_Y^2) \end{bmatrix}$$

$$Trace[\mathcal{E}\mathcal{E}^T] = 2k^2(T_X^2+T_Y^2+T_Z^2) = 2k^2\|t\|^2$$

Reconstruction up to a Scale Factor

$$\mathcal{E} = k[t_\times]R$$

$$\mathcal{E}\mathcal{E}^T = k^2[t_\times]RR^T[t_\times]^T = k^2[t_\times][t_\times]^T = \begin{bmatrix} k^2(T_Y^2+T_Z^2) & -k^2T_XT_Y & -k^2T_XT_Z \\ -k^2T_XT_Y & k^2(T_X^2+T_Z^2) & -k^2T_YT_Z \\ -k^2T_XT_Z & -k^2T_YT_Z & k^2(T_X^2+T_Y^2) \end{bmatrix}$$

$$Trace[\mathcal{E}\mathcal{E}^T] = 2k^2(T_X^2+T_Y^2+T_Z^2) = 2k^2\|t\|^2$$

$$\frac{\mathcal{E}}{|k|\|t\|} = \mathrm{sgn}(k)\frac{[t_\times]}{\|t\|}R = \mathrm{sgn}(k)\left[\left(\frac{t}{\|t\|}\right)_\times\right]R$$

Reconstruction up to a Scale Factor

$$\mathcal{E} = k[t_\times]R$$

$$\mathcal{E}\mathcal{E}^T = k^2[t_\times]RR^T[t_\times]^T = k^2[t_\times][t_\times]^T = \begin{bmatrix} k^2\left(T_Y^2 + T_Z^2\right) & -k^2 T_X T_Y & -k^2 T_X T_Z \\ -k^2 T_X T_Y & k^2\left(T_X^2 + T_Z^2\right) & -k^2 T_Y T_Z \\ -k^2 T_X T_Z & -k^2 T_Y T_Z & k^2\left(T_X^2 + T_Y^2\right) \end{bmatrix}$$

$$Trace[\mathcal{E}\mathcal{E}^T] = 2k^2\left(T_X^2 + T_Y^2 + T_Z^2\right) = 2k^2\|t\|^2$$

$$\frac{\mathcal{E}}{|k|\|t\|} = \operatorname{sgn}(k)\frac{[t_\times]}{\|t\|}R = \operatorname{sgn}(k)\left[\left(\frac{t}{\|t\|}\right)_\times\right]R = \operatorname{sgn}(k)[\hat{t}_\times]R$$

Reconstruction up to a Scale Factor

$$\mathcal{E} = k[t_\times]R$$

$$\mathcal{E}\mathcal{E}^T = k^2[t_\times]RR^T[t_\times]^T = k^2[t_\times][t_\times]^T = \begin{bmatrix} k^2(T_Y^2+T_Z^2) & -k^2T_XT_Y & -k^2T_XT_Z \\ -k^2T_XT_Y & k^2(T_X^2+T_Z^2) & -k^2T_YT_Z \\ -k^2T_XT_Z & -k^2T_YT_Z & k^2(T_X^2+T_Y^2) \end{bmatrix}$$

$$Trace[\mathcal{E}\mathcal{E}^T] = 2k^2(T_X^2+T_Y^2+T_Z^2) = 2k^2\|t\|^2$$

$$\frac{\mathcal{E}}{|k|\|t\|} = \mathrm{sgn}(k)\frac{[t_\times]}{\|t\|}R = \mathrm{sgn}(k)\left[\left(\frac{t}{\|t\|}\right)_\times\right]R = \mathrm{sgn}(k)[\hat{t}_\times]R = \hat{E}$$

Reconstruction up to a Scale Factor

$$\mathcal{E} = k[t_\times]R$$

$$\mathcal{E}\mathcal{E}^T = k^2[t_\times]RR^T[t_\times]^T = k^2[t_\times][t_\times]^T = \begin{bmatrix} k^2(T_Y^2+T_Z^2) & -k^2T_XT_Y & -k^2T_XT_Z \\ -k^2T_XT_Y & k^2(T_X^2+T_Z^2) & -k^2T_YT_Z \\ -k^2T_XT_Z & -k^2T_YT_Z & k^2(T_X^2+T_Y^2) \end{bmatrix}$$

$$Trace[\mathcal{E}\mathcal{E}^T] = 2k^2(T_X^2+T_Y^2+T_Z^2) = 2k^2\|t\|^2$$

$$\frac{\mathcal{E}}{|k|\|t\|} = \operatorname{sgn}(k)\frac{[t_\times]}{\|t\|}R = \operatorname{sgn}(k)\left[\left(\frac{t}{\|t\|}\right)_\times\right]R = \operatorname{sgn}(k)[\hat{t}_\times]R = \hat{E}$$

$$\hat{E}\hat{E}^T = [\hat{t}_\times][\hat{t}_\times]^T$$

Reconstruction up to a Scale Factor

$$\mathcal{E} = k[t_\times]R$$

$$\mathcal{E}\mathcal{E}^T = k^2[t_\times]RR^T[t_\times]^T = k^2[t_\times][t_\times]^T = \begin{bmatrix} k^2(T_Y^2+T_Z^2) & -k^2T_XT_Y & -k^2T_XT_Z \\ -k^2T_XT_Y & k^2(T_X^2+T_Z^2) & -k^2T_YT_Z \\ -k^2T_XT_Z & -k^2T_YT_Z & k^2(T_X^2+T_Y^2) \end{bmatrix}$$

$$Trace[\mathcal{E}\mathcal{E}^T] = 2k^2(T_X^2+T_Y^2+T_Z^2) = 2k^2\|t\|^2$$

$$\frac{\mathcal{E}}{|k|\|t\|} = \mathrm{sgn}(k)\frac{[t_\times]}{\|t\|}R = \mathrm{sgn}(k)\left[\left(\frac{t}{\|t\|}\right)_\times\right]R = \mathrm{sgn}(k)[\hat{t}_\times]R = \hat{E}$$

$$\hat{E}\hat{E}^T = [\hat{t}_\times][\hat{t}_\times]^T = \begin{bmatrix} 1-\hat{T}_X^2 & -\hat{T}_X\hat{T}_Y & -\hat{T}_X\hat{T}_Z \\ -\hat{T}_X\hat{T}_Y & 1-\hat{T}_Y^2 & -\hat{T}_Y\hat{T}_Z \\ -\hat{T}_X\hat{T}_Z & -\hat{T}_Y\hat{T}_Z & 1-\hat{T}_Z^2 \end{bmatrix}$$

Reconstruction up to a Scale Factor

$$\hat{E} = \begin{bmatrix} \hat{E}_1^T \\ \hat{E}_2^T \\ \hat{E}_3^T \end{bmatrix} \qquad R = \begin{bmatrix} R_1^T \\ R_2^T \\ R_3^T \end{bmatrix}$$

Let $w_i = \hat{E}_i \times \hat{t},\ i \in \{1,2,3\}$

It can be proved that

$$R_1 = w_1 + w_2 \times w_3$$
$$R_2 = w_2 + w_3 \times w_1$$
$$R_3 = w_3 + w_1 \times w_2$$

Reconstruction up to a Scale Factor

We have two choices of **t**, (**t^+** and **t^-)** because of sign ambiguity
and two choices of **E,** (E^+ and E^-).

This gives us four pairs of translation vectors and rotation matrices.

Reconstruction up to a Scale Factor

Given $\hat{E}$ and $\hat{t}$

1. Construct the vectors **w**, and compute R
2. Reconstruct the Z and Z' for each point
3. If the signs of Z and Z' of the reconstructed points are
   a) both negative for some point, change the sign of $\hat{t}$ and go to step 2.
   b) different for some point, change the sign of each entry of $\hat{E}$ and go to step 1.
   c) both positive for all points, exit.

$$Z = f \frac{\left(x' R'_3 - f' R'_1\right)^T t}{\left(x' R'_3 - f' R'_1\right)^T p}$$

$$Z' = -f' \frac{\left(x R_3 - f R_1\right)^T (t)}{\left(x R_3 - f R_1\right)^T p'}$$

3D Reconstruction

[Trucco pp. 161]

- Three cases:
  - a) intrinsic and extrinsic parameters known: Solve reconstruction by triangulation: ray intersection
  - b) only intrinsic parameters known: estimate essential matrix E up to scaling
  - c) intrinsic and extrinsic parameters not known: estimate fundamental matrix F, reconstruction up to global, projective transformation

Run Example

Demo for stereo reconstruction:

http://mitpress.mit.edu/e-journals/Videre/001/articles/Zhang/CalibEnv/CalibEnv.html